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On a Centrality Maximization Game

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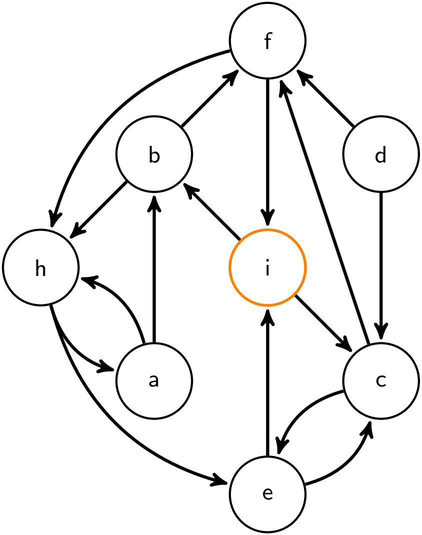
Workshop on Network Dynamics in the Social, Economic, and Financial Sciences

Internet can be modeled by:

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, P)$$

$\mathcal{V}$ set of webpages
$\mathcal{E}$ set of hyperlinks
P normalized weight matrix

Bonacich Centrality



How to measure the **importance** π_i of a position?

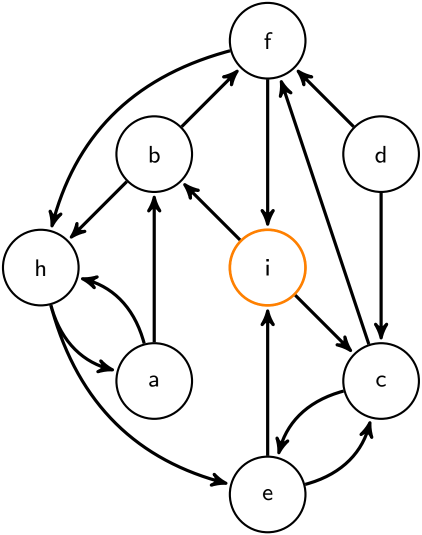
$$\pi = \beta P'\pi + (1-\beta)\eta,$$

where $\sum_i \eta_i = 1$.

Bonacich centrality

Applications: **visibility** of a webpage on the internet.

Centrality Maximization Game

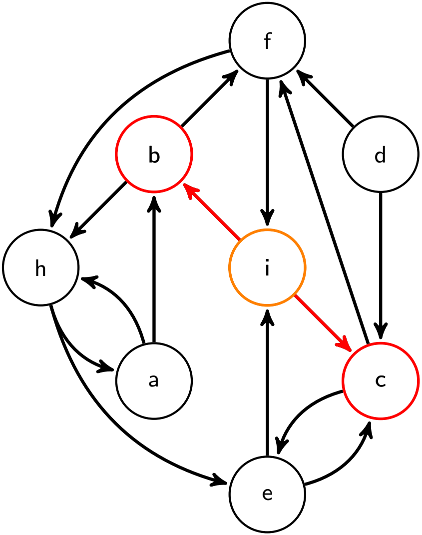


$\Gamma(\mathcal{V}, \beta, \eta, m) = (\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}})$

- $\mathcal{V}=$ {players}

$x \Rightarrow \mathcal{G}(x)$.

Centrality Maximization Game



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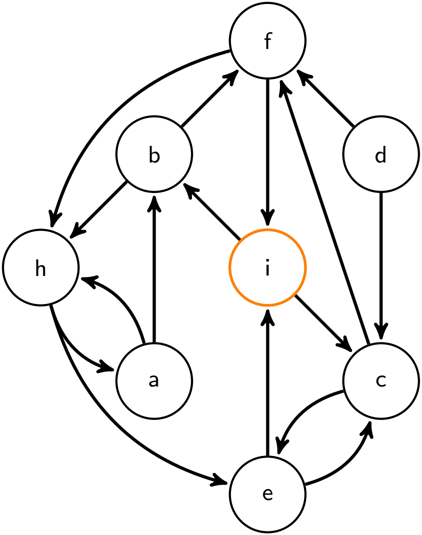
- $\mathcal{V}$= {players}
- action $x_i \in \mathcal{A}_i$ of node i is the set of m nodes node i points to.

$x \Rightarrow \mathcal{G}(x)$.

Centrality Maximization Game



$\Gamma(\mathcal{V}, \beta, \eta, m) = (\mathcal{V}, \{\mathcal{A}_i\}_{i\in\mathcal{V}}, \{u_i\}_{i\in\mathcal{V}})$

- $\mathcal{V}$= {players}
- action $x_i \in \mathcal{A}_i$ of node i is the set of m nodes node i points to.
- utility u_i of node i is the Bonacich centrality.

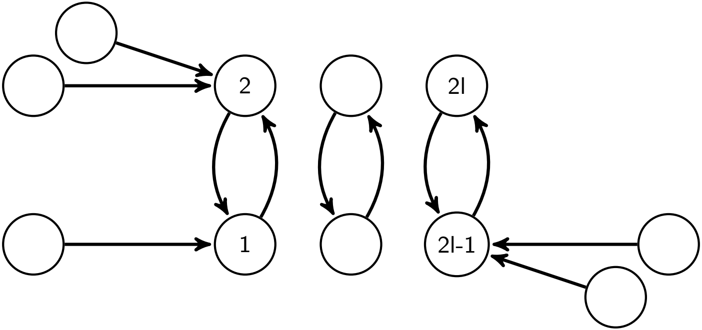
$x \Rightarrow \mathcal{G}(x)$.

Results for m=1: Nash equilibria

x^* is a Nash equilibrium $\Leftrightarrow \mathcal{G}(x^*)$ is of type $C_2^{l,r}$, where $2l + r = |\mathcal{V}|$:

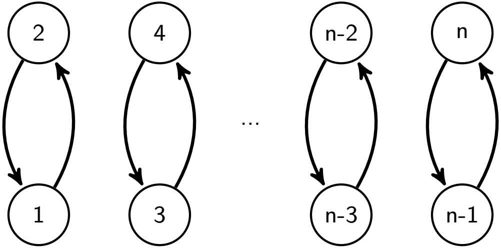
- l number of 2-cliques;
- r nodes with in-degree equal to zero pointing at a 2-clique.

Example:

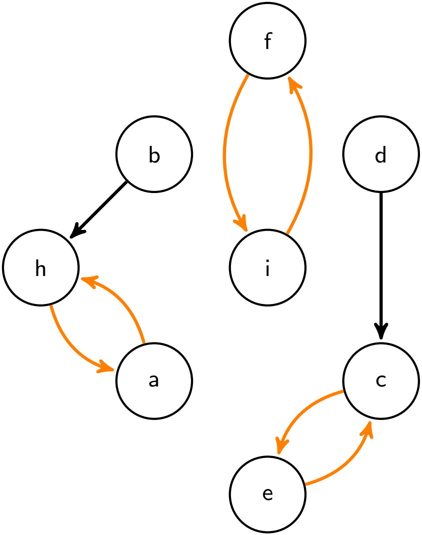


Results for m=1: strict Nash equilibria

- **If** n **is even**, x^* is a strict Nash equilibrium $\Leftrightarrow \mathcal{G}(x^*)$ is of type $C_2^{n/2,0}$.
- **If** n **is odd**, there are no strict Nash equilibria.



Results for m=1: ordinal potential

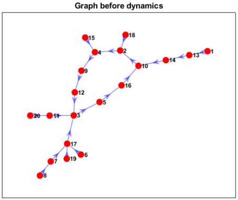


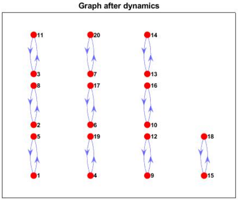
- Function $\Psi : \mathcal{A} \to \mathbb{N}$ counting the number of 2-cliques is an **ordinal potential**;

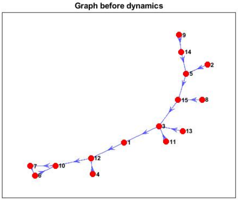
$\Psi = 3$

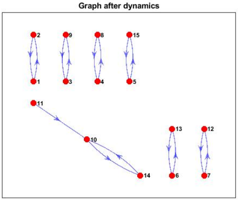
Results for m=1: Best Response Dynamics

The best response dynamics of $\Gamma(\mathcal{V}, \beta, 1)$ always converges to a Nash equilibrium.









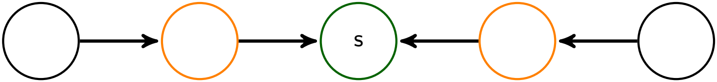
The **limit set** $\mathcal{N}^*$ is:

- set of $C_2^{n/2,0}$, if n is even;
- set of $C_2^{(n-1)/2,1}$, if n is odd;

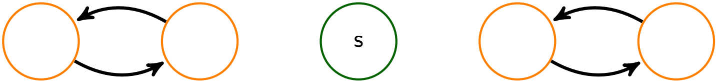
Idea of the proofs

Best Response Function

- if $N_s^-(x) \neq \emptyset \;\Rightarrow\; B_s(x_{-s}) = \{j \mid j \in N_s^-(x)\}$



- if $N_s^-(x) = \emptyset \;\Rightarrow\; B_s(x_{-s}) = \{j \mid j \in \mathcal{V}\}$.

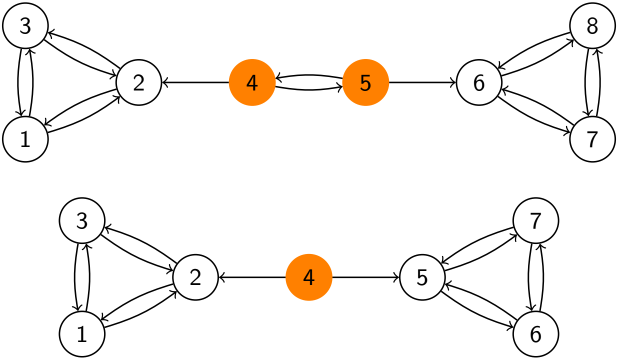


where $N_s^-(x)$ set of in-neighbors of s.

Results for m=2: Nash equilibria

If x^* is a Nash equilibrium:

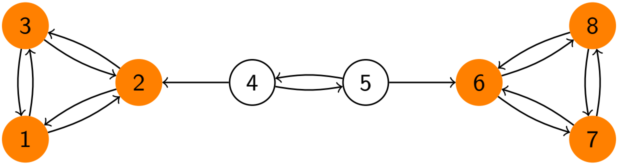
- every connected component of $G(x)$ is either a sink or a source. (if not isolated)
- every source component is a singleton or a 2-clique

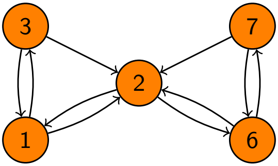


Results for m=2: Nash equilibria

If x^* is a Nash equilibrium:

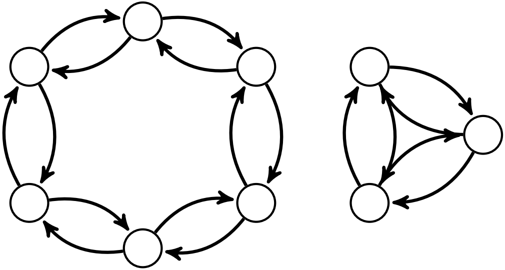
- every connected component of $G(x)$ is either a sink or a source. (if not isolated)
- every source component is a singleton or a 2-clique
- every sink component is either a cycle or the Butterfly graph





Results for m=2: Strict Nash equilibria

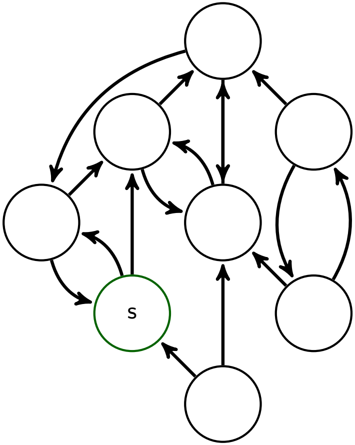
x^* is a strict Nash equilibrium $\Leftrightarrow G(x^*)$ is undirected.



Idea of the proofs

Characteristics of the best response set

if $|N_s^{-2}(x)| \geq 2$



where $N_s^{-2}(x) = \{ j \mid dist_x(j, s) \leq 2 \}$.

Idea of the proofs

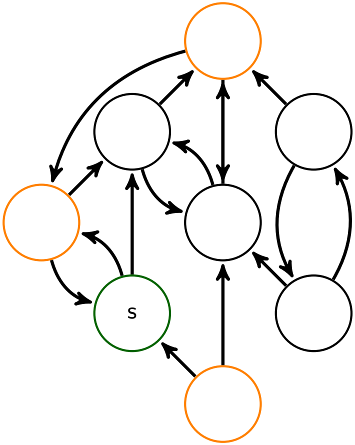
Characteristics of the best response set

if $|N_s^{-2}(x)| \geq 2$

$$\Downarrow$$

$B_s(x_{-s}) \subseteq N_s^{-2}(x)$

where $N_s^{-2}(x) = \{\, j \mid dist_x(j,s) \leq 2\}$.



Conclusions and Next Steps

Results:

- The best strategy to maximize the Bonacich centrality is to **act locally**.

- Strict Nash equilibria are all and only **undirected graphs** if $m = 1$ or $m = 2$.
- Existence of an **ordinal potential** for $m = 1$.

Next steps:

- Generalization to any m.

Thank you for your attention!