POLITECNICO DI TORINO

On graphicality and decomposition of games

Laura Arditti,
joint work with Giacomo Como and Fabio Fagnani

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INTRODUCTION

- Graphical games: games equipped with a network structure
  - A graph specifies the pattern of dependence of utilities
- Model for the emergence of global phenomena
  - in socio-economic networks
  - in engineering and computer science
- A general theory is still missing
  - how graphicality of games reflects on their properties?

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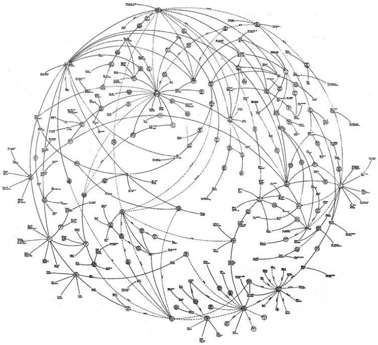

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GAMES

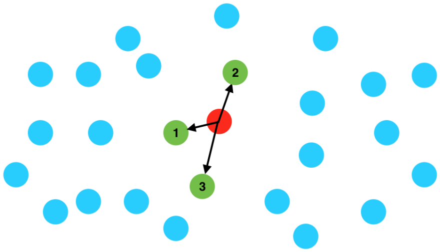
- Strategic form games:
- $\mathcal{V}$ players
  $\forall i \in \mathcal{V}, A^i$ actions of i
  $\mathcal{X} = \prod_{i \in \mathcal{V}} A^i$ strategies
- $\forall i \in \mathcal{V}$, $u_i : \mathcal{X} \to \mathbb{R}$ utility of i.
- $\Gamma(\mathcal{V}, \mathcal{X})$: games with players $\mathcal{V}$ and strategies $\mathcal{X}$, identified by the utility vector u.
- $x, y \in \mathcal{X}$, $i \in \mathcal{V}$. We say that x and y are i-comparable, if $x_j = y_j$ for every $j \neq i$ and $x_i \neq y_i$.
  $\mathcal{X}^{(2)} \subset \mathcal{X} \times \mathcal{X}$ pairs of comparable strategies.

GRAPHICAL GAMES

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ digraph. $u \in \Gamma(\mathcal{V}, \mathcal{X})$ is a $\mathcal{G}$-game if for every $x, y \in \mathcal{X}$

$$x_j = y_j \; \forall j \in N(i)^{+} \cup \{i\} \;\Rightarrow\; u_i(x) = u_i(y) \, .$$

  - Every game is graphical on the complete graph on $\mathcal{V}$
  - $\mathcal{G}(u)$ is the minimal graph of u



- Pairwise graphical games
  - graphicality is a design property
  - undirected weighted graph $\mathcal{G} = (\mathcal{V}, W, \mathcal{E})$. $\forall \{i,j\} \in \mathcal{E}$, players i and j are involved in a 2-player game $u_{\{i,j\}}$, with utilities u_{ij} and u_{ji}.
  - $u \in \Gamma_{\mathcal{G}}(\mathcal{V}, \mathcal{X})$ is a pairwise graphical $\mathcal{G}$-game if

$$u_i(x) = \sum_{j \in \mathcal{V}} W_{ij} u_{ij}(x_i, x_j) \qquad \forall x \in \mathcal{X} \, ,$$

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STRATEGIC EQUIVALENCE

- $u \in \Gamma(\mathcal{V}, \mathcal{X})$ is a non-strategic game ($u \in \mathcal{N}$) if $\forall i \in \mathcal{V}$ and any $(x, y) \in \mathcal{X}^{(2)}$ i-comparable,
$$u_i(x) - u_i(y) = 0\,.$$
- Games in $\mathcal{N}^{\perp}$ are called normalized games as they satisfy the normalization condition:
$$\forall i \in \mathcal{V}, \forall x \in \mathcal{X}, \qquad \sum_{y:(y,x) \text{ are } i\text{-comp.}} u_i(y) = 0$$
  - u, $v \in \Gamma(\mathcal{V}, \mathcal{X})$ are strategically equivalent if $u - v \in \mathcal{N}$.
  - in every strategic equivalence class $[u]$, $u \in \Gamma(\mathcal{V}, \mathcal{X})$, there exists exactly one normalized member u^{norm}.

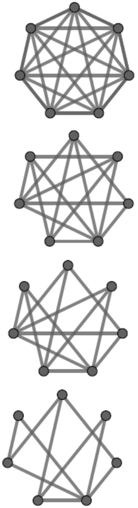
GRAPHICALITY OF STRATEGICALLY EQUIVALENT GAMES

- Strategic equivalence does not preserve graphicality.

Theorem

Let $u \in \Gamma(\mathcal{V}, \mathcal{X})$. There exist $u^* \in [u]$ such that $\mathcal{G}(u^*)$ is minimal, i.e. $\mathcal{G}(u^*) \subseteq \mathcal{G}(v)$ for every $v \in [u]$.
Moreover, a possible choice for u^* is the normalized game u^{norm}.

- The minimal graph $\mathcal{G}([u])$ represents the minimal topological complexity needed to represent a game in $[u]$.



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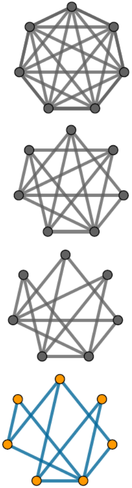

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POTENTIAL AND HARMONIC GAMES

- $u \in \Gamma(\mathcal{V}, \mathcal{X})$ is a potential game if there exists a function $\phi \in \mathbb{R}^{\mathcal{X}}$, called potential, such that $\forall i \in \mathcal{V}$ and any two strategies x and y that are i-comparable

$$u_i(x) - u_i(y) = \phi(x) - \phi(y) \,, \quad \forall (x, y) \in \mathcal{X}^{(2)}$$

- $u \in \Gamma(\mathcal{V}, \mathcal{X})$ is an harmonic game if for every strategy $x \in \mathcal{X}$

$$\sum_{i \in \mathcal{V}} \sum_{y:(y,x) \text{ are } i\text{-comp.}} u_i(x) - u_i(y) = 0$$

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DECOMPOSITION OF GAMES

Theorem (Candogan, Menache, Ozdaglar, Parrilo, 2010)

The space of games can be decomposed as

$$\Gamma(\mathcal{V}, \mathcal{X}) = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$$

where $\oplus$ denotes the direct sum. $\mathcal{P} = \mathcal{N}^{\perp} \cap P$ is the space of normalized potential games, $\mathcal{N}$ is the space of non-strategic games, $\mathcal{H} = \mathcal{N}^{\perp} \cap H$ is the space of normalized harmonic games.

- Classical decomposition does not take into account graphical structure of games.
- Main result: how graphicality interacts with the decomposition of games.

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DECOMPOSITION OF GRAPHICAL GAMES

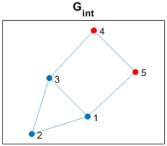
$u \in \Gamma_{\mathcal{G}}(\mathcal{X}, \mathcal{V})$. On which graph are its component graphical? What is the relation with $\mathcal{G}$?

- Pairwise graphical games $\rightarrow$ classical decomposition can be exploited

Theorem

Let u be a pairwise $\mathcal{G}$-game with decomposition $u = u_{\mathcal{P}} + u_{\mathcal{H}} + u_{\mathcal{N}}$. Then

- $\mathcal{G}(u_{\mathcal{P}})$ contains $\{i,j\} \in \mathcal{E}$ iff $u_{\{i,j\}}$ is not purely harmonic,
- $\mathcal{G}(u_{\mathcal{H}})$ contains $\{i,j\} \in \mathcal{E}$ iff $u_{\{i,j\}}$ is not purely potential,
- $\mathcal{G}(u_{\mathcal{N}})$ contains $\{i,j\} \in \mathcal{E}$ iff $u_{\{i,j\}}$ is not normalized.



- $\mathcal{G}(u_{\mathcal{P}})$,$\mathcal{G}(u_{\mathcal{H}})$,$\mathcal{G}(u_{\mathcal{N}})$ are subgraphs of $\mathcal{G}$.
  - decomposition does not create any link between players not directly interacting in the original game.

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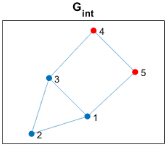

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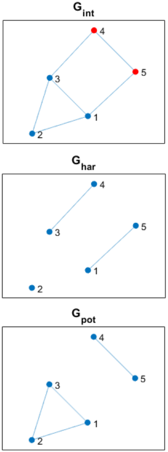


DECOMPOSITION OF GRAPHICAL GAMES

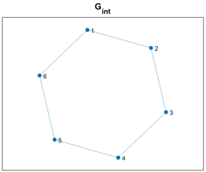
- Non pairwise graphical games → graphicality and decomposition interact in a complex fashion

Theorem 2

Every game $u \in \Gamma_{\mathcal{G}}(\mathcal{V}, \mathcal{X})$ can be decomposed as $u = u_{\mathcal{P}} + u_{\mathcal{H}} + u_{\mathcal{N}}$ where

- the normalized potential component $u_{\mathcal{P}}$ is a $\mathcal{G}^{\triangle}$-game
- the normalized harmonic component $u_{\mathcal{H}}$ is a $\mathcal{G}^{\triangle}$-game
- the non-strategic component $u_{\mathcal{N}}$ is a $\mathcal{G}$-game

- $\mathcal{G}^{\triangle}$: undirected graph with nodes $\mathcal{V}$ and links among players belonging to a common out-neighbourhood in $\mathcal{G}$.
    - Hidden strategic interactions have short range.



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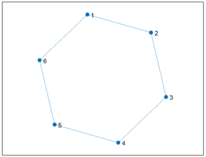
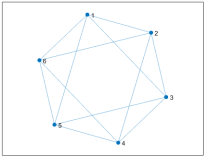
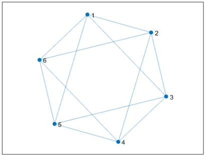


G_{pot}



G_{har}



GAME FLOWS

- The space of flows is

$$F\ell = \left\{ F \in \mathbb{R}^{\mathcal{X}^{(2)}} \mid F(x,y) = -F(y,x),\ \forall (x,y) \in \mathcal{X}^{(2)} \right\}$$

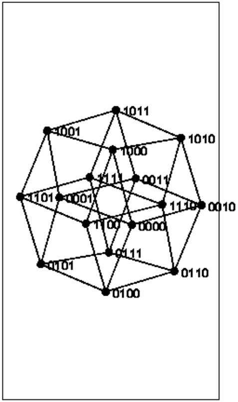
  - Flows are defined on the edges of the strategy graph $\mathcal{G}_{str} = (\mathcal{X}, \mathcal{X}^{(2)})$.

- $D : \Gamma(\mathcal{V}, \mathcal{X}) \to F\ell$ maps the game $u \in \Gamma(\mathcal{V}, \mathcal{X})$ to the flow $Du = F \in F\ell$ s.t.

$$F(x,y) = u_i(y) - u_i(x) \quad \forall (x,y) \in \mathcal{X}^{(2)}$$

where i is the only player s.t. x and y are i-comparable.

  - Flow characterization of potentiality, harmonicity and graphicality.
    - This allows us studying graphical games by analysing their flows.



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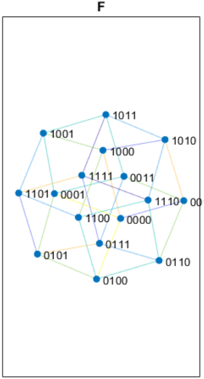

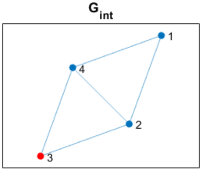

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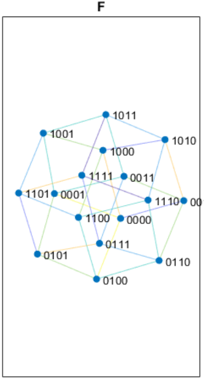
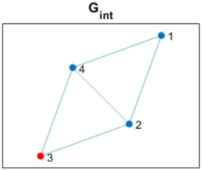
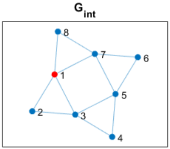




EXAMPLE



- $\mathcal{G}$ represents social interactions of players $\mathcal{V}$
- $\mathcal{A}^i = \{0, 1\}$, $\forall i \in \mathcal{V}$: players decide of acquiring (action 1) or not acquiring (0) some good
  - players $i \neq 1$ have imitative behaviour $\rightarrow$ majority game

    $$u_i(a_i, x_{N(i)}) = |\{j \in N(i) : x_j = a_i\}|, \qquad a_i = 0, 1$$

  - player 1 $\rightarrow$ public good game

    $$\begin{aligned} u_1(1, x_{N(1)}) &= 1 - c \\ u_1(0, x_{N(1)}) &= 1 \qquad \text{if } x_j = 1 \text{ for some } j \in N(1) \\ u_1(0, x_{N(1)}) &= 0 \qquad \text{if } x_j = 0 \text{ for all } j \in N(1) \end{aligned}$$

- Local perturbation of a potential game
  - locality is preserved in the decomposition.

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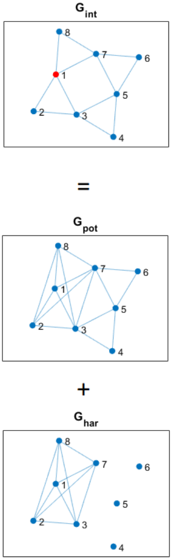

FUTURE WORK

- Refinement of the results
  - Separable graphical games
- Interpretation of the results
  - Role of hidden strategic interactions
- Graphical potential games and Markov Random Fields
  - Decomposition of the potential
- Robustness analysis of games
  - Properties of perturbations of potential games

Thank you for the attention

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