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Initial-value problem for the two-dimensional growing wake

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Introduction and Outline

- A general three-dimensional initial-value perturbation problem is presented to study the linear stability of the parallel and weakly non-parallel wake (Belan & Tordella, 2002 Zamm; Tordella & Belan, 2003 PoF);

- Arbitrary three-dimensional perturbations physically in terms of the vorticity are imposed (Blossey, Criminale & Fisher, submitted 2006 JFM);

- Investigation of both the early transient as well as the asymptotics fate of any disturbances (Criminale, Jackson & Lasseigne, 1995 JFM);

- Numerical resolution by method of lines of the governing PDEs after Fourier transform in streamwise and spanwise directions;

- Some results and comparison with recent normal modes theory analyses (Tordella, Scarsoglio & Belan, 2006 PoF; Belan & Tordella, 2006 JFM).

Formulation

- Linear, three-dimensional perturbative equations (non-dimensional quantities with respect to the base flow and spatial scales);
- Viscous, incompressible, constant density fluid;
- Base flow: - parallel $U(y) = 1 - \mathrm{sech}^2(y)$
  - 2D asymptotic Navier-Stokes expansion (Belan & Tordella, 2003 PoF) parametric in x_0

$$U(x_0, y) = 1 - x_0^{-1/2} C_1 \mathrm{e}^{-\frac{R}{4}\frac{y^2}{x_0}}$$
$$-x_0^{-1} \frac{A^2}{2} \mathrm{e}^{-\frac{R}{4}\frac{y^2}{x_0}} \left\{ \mathrm{C}_2\, {}_1\mathrm{F}_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{R}{4}\frac{y^2}{x_0}\right) + \mathrm{e}^{-\frac{R}{4}\frac{y^2}{x_0}} + \frac{\sqrt{\pi R}}{2}\frac{y}{\sqrt{x_0}} \mathrm{erf}\left(\frac{\sqrt{R}}{2}\frac{y}{\sqrt{x_0}}\right) \right\}$$

$$\begin{cases} \nabla^2 \tilde{v} = \tilde{\Gamma} \\ \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\tilde{\Gamma} - \frac{d^2U}{dy^2}\frac{\partial \tilde{v}}{\partial x} = \frac{1}{R}\nabla^2 \tilde{\Gamma} \\ \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\tilde{\omega}_y + \frac{dU}{dy}\frac{\partial \tilde{v}}{\partial z} = \frac{1}{R}\nabla^2 \tilde{\omega}_y \end{cases}$$

$$\tilde{\omega}_y = \frac{\partial \tilde{u}}{\partial z} - \frac{\partial \tilde{w}}{\partial x}$$

$$\tilde{\Gamma} = \frac{\partial \tilde{\omega}_z}{\partial x} - \frac{\partial \tilde{\omega}_x}{\partial z}$$

disturbance velocity $(\tilde{u}(t,x,y,z), \tilde{v}(t,x,y,z), \tilde{w}(t,x,y,z))$

disturbance vorticity $(\tilde{\omega}_x(t,x,y,z), \tilde{\omega}_y(t,x,y,z), \tilde{\omega}_z(t,x,y,z))$

• Moving coordinate transform $\xi = x - U_o t$ (Criminale & Drazin, 1990 Stud. Appl. Maths), with $U_o = U(y \rightarrow \infty)$

• Fourier transform in ξ and z directions: $\hat{f}(y,t;\alpha,\gamma) = \int\int_{-\infty}^{+\infty} \tilde{f} e^{i\alpha\xi + i\gamma z} d\xi dz$

$$\begin{cases} \dfrac{\partial^2 \hat{v}}{\partial y^2} - k^2 \hat{v} = \hat{\Gamma} \\ \dfrac{\partial \hat{\Gamma}}{\partial t} = -ikcos(\phi)(U - U_0)\hat{\Gamma} + ikcos(\phi)\dfrac{\partial^2 U}{\partial y^2}\hat{v} + \dfrac{1}{R}(\dfrac{\partial^2 \hat{\Gamma}}{\partial y^2} - k^2 \hat{\Gamma}) \\ \dfrac{\partial \hat{\omega}_y}{\partial t} = -ikcos(\phi)(U - U_0)\hat{\omega}_y - iksin(\phi)\dfrac{\partial U}{\partial y}\hat{v} + \dfrac{1}{R}(\dfrac{\partial^2 \hat{\omega}_y}{\partial y^2} - k^2 \hat{\omega}_y) \end{cases}$$

$\alpha = k\ cos(\Phi)$ wavenumber in ξ-direction

$\gamma = k\ sin(\Phi)$ wavenumber in z-direction

$\Phi = tan^{-1}(\gamma/\alpha)$ angle of obliquity

$k = (\alpha^2 + \gamma^2)^{1/2}$ polar wavenumber.

Numerical solutions

• Initial disturbances are periodic and bounded in the free stream:

$$\widetilde{\omega}_y(y, t=0) = 0 \qquad \begin{cases} \widetilde{v}(y, t=0) = e^{-y^2} sin(\beta_0 y) \\ \quad \text{or} \\ \widetilde{v}(y, t=0) = e^{-y^2} cos(\beta_0 y) \end{cases}$$

• Numerical resolution by the method of lines:

- spatial derivatives computed using compact finite differences;
- time integration with an adaptative, multistep method (variable order Adams-Bashforth-Moulton PECE solver), Matlab function ode113.

Total kinetic energy of the perturbation is defined (Blossey, Criminale & Fisher, submitted 2006 JFM) as:

$$E(t) = \int_x \int_y \int_z \frac{1}{2}(\widetilde{u}^2 + \widetilde{v}^2 + \widetilde{w}^2)dxdydz =$$

$$\int_k \int_\phi \frac{1}{2k^2} \int_y (|\partial\widehat{v}/\partial y|^2 + k^2|\widehat{v}|^2 + |\widehat{\omega}_y|^2)dyd\phi dk$$

$$ke(t;k,\phi) = k^2E(t) = \frac{1}{2}\int_y (|\frac{\partial \widehat{v}}{\partial y}|^2 + k^2|\widehat{v}|^2 + |\widehat{\omega}_y|^2)dy$$ energy density

The growth function *G* defined in terms of the normalized energy density

$$G(t;k,\phi) = \frac{ke(t;k,\phi)}{ke(t=0;k,\phi)}$$

can effectively measure the growth of the energy at time *t*, for a given initial condition at $t = 0$.
Considering that the amplitude of the disturbance is proportional to $\widetilde{v} \approx e^{rt}$, the temporal growth rate can be defined (Lasseigne, et al., 1999 JFM) as

$$r = \frac{log|E(t)|}{2t}$$

For configurations that are asymptotically unstable, the equations are integrated forward in time beyond the transient until the growth rate *r* asymptotes to a constant value (for example $dr/dt < \varepsilon = 10^{-5}$).
⟶ Comparison with results by non parallel normal modes analyses (Tordella, Scarsoglio & Belan, 2006 PoF; Belan & Tordella, 2006 JFM) can be done.

Order zero theory. Homogeneous Orr-Sommerfeld equation (parametric in x).

$$\begin{cases} \mathcal{A}\varphi_0 = \sigma_0 \mathcal{B}\varphi_0 \\ \varphi_0 \to 0, \ |y| \to \infty \\ \partial_y \varphi_0 \to 0, \ |y| \to \infty \end{cases}$$

$$\mathcal{A} = (\partial_y^2 - h_0^2)^2 - ih_0 R[u_0(\partial_y^2 - h_0^2) - \partial_y^2 u_0]$$

$$\mathcal{B} = -iR(\partial_y^2 - h_0^2)$$

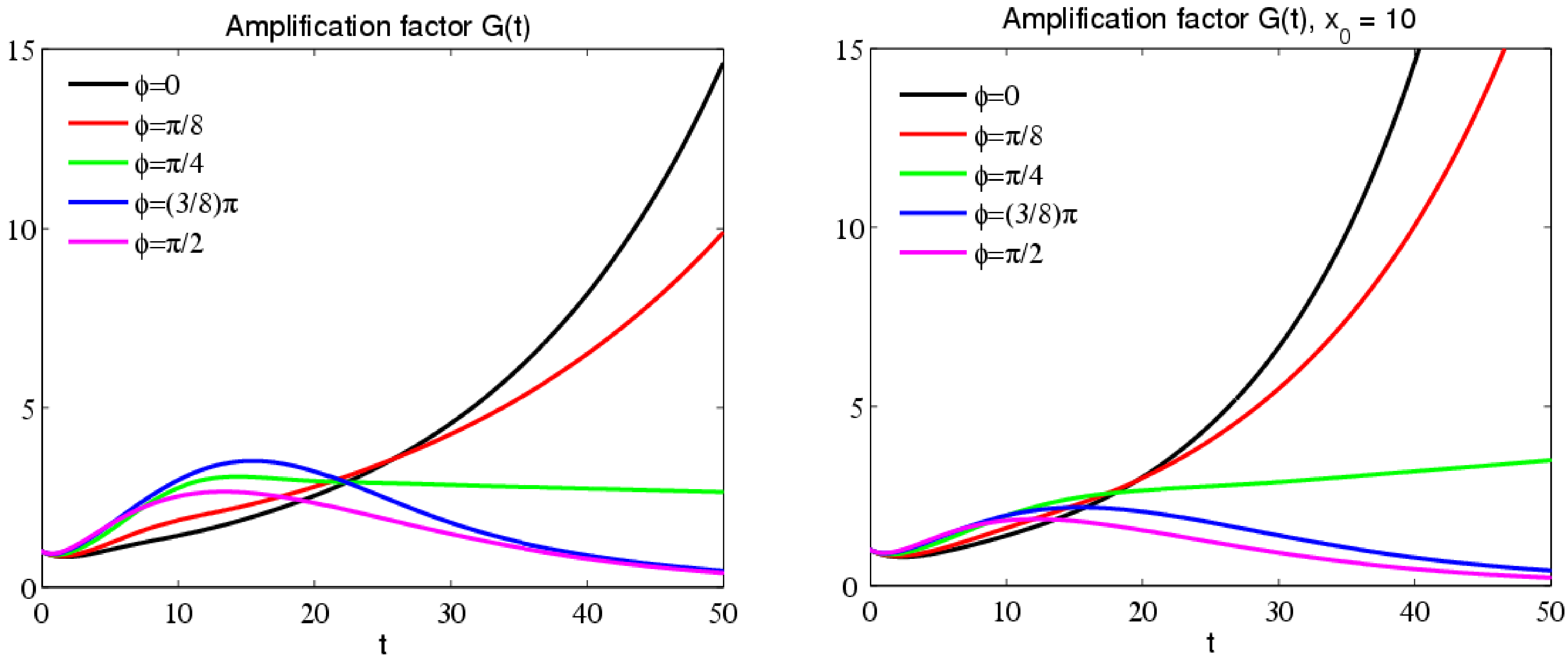
By numerical solution $\longrightarrow$ eigenfunctions φ_0 and a discrete set of eigenvalues σ_{on}

First order theory. Non homogeneous Orr-Sommerfeld equation (x parameter).

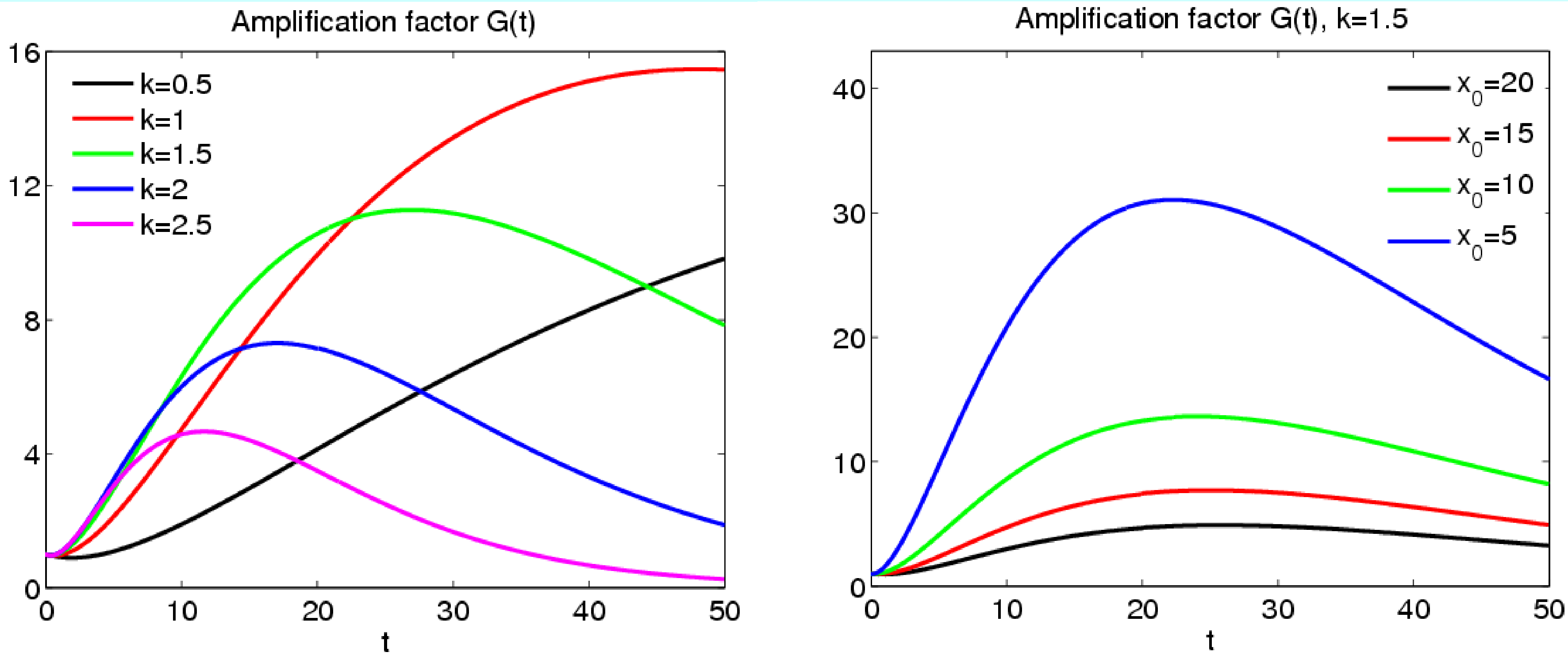
$$\begin{cases} \mathcal{A}\varphi_1 = \sigma_0 \mathcal{B}\varphi_1 + \mathcal{M}\varphi_0 \\ \varphi_1 \to 0, \ |y| \to \infty \\ \partial_y \varphi_1 \to 0, \ |y| \to \infty \end{cases}$$

$\mathcal{M}$ is related to base flow and it considers non-parallel effects through transverse velocity presence

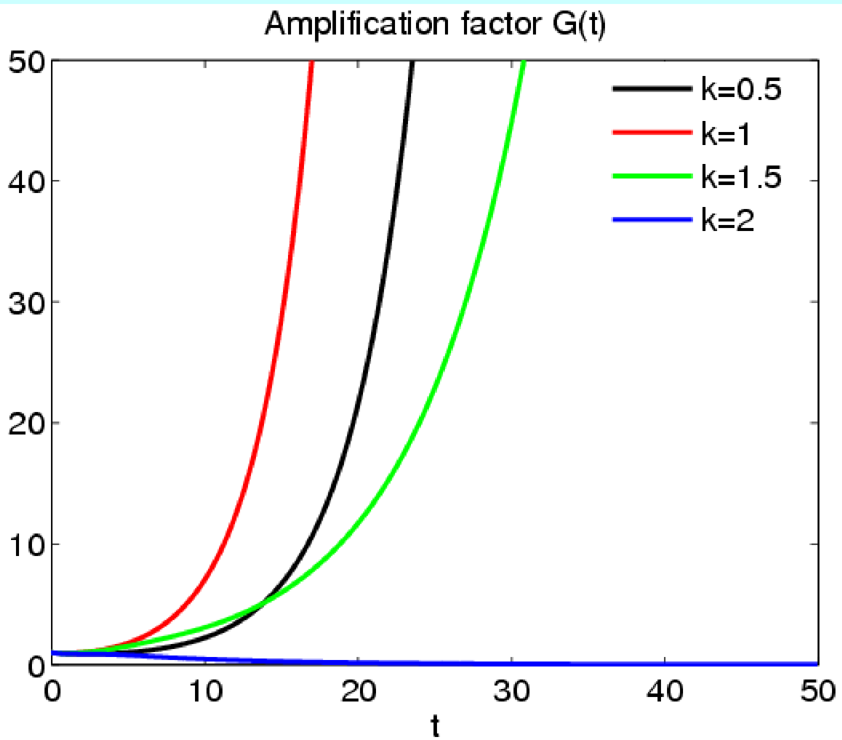
$$\begin{aligned} \mathcal{M} \;=\; & \{ \left[R(2h_0\sigma_0 - 3h_0^2 u_0 - \partial_y^2 u_0) + 4ih_0^3 \right] \partial_{x_1} \\ & + (Ru_0 - 4ih_0)\partial_{xyy}^3 - Rv_1(\partial_y^3 - h_0^2 \partial_y) + R\partial_y^2 v_1 \partial_y \\ & + ih_0 R \left[u_1(\partial_y^2 - h_0^2) - \partial_y^2 u_1 \right] + R(\partial_y^2 - h_0^2)\partial_t \} \end{aligned}$$

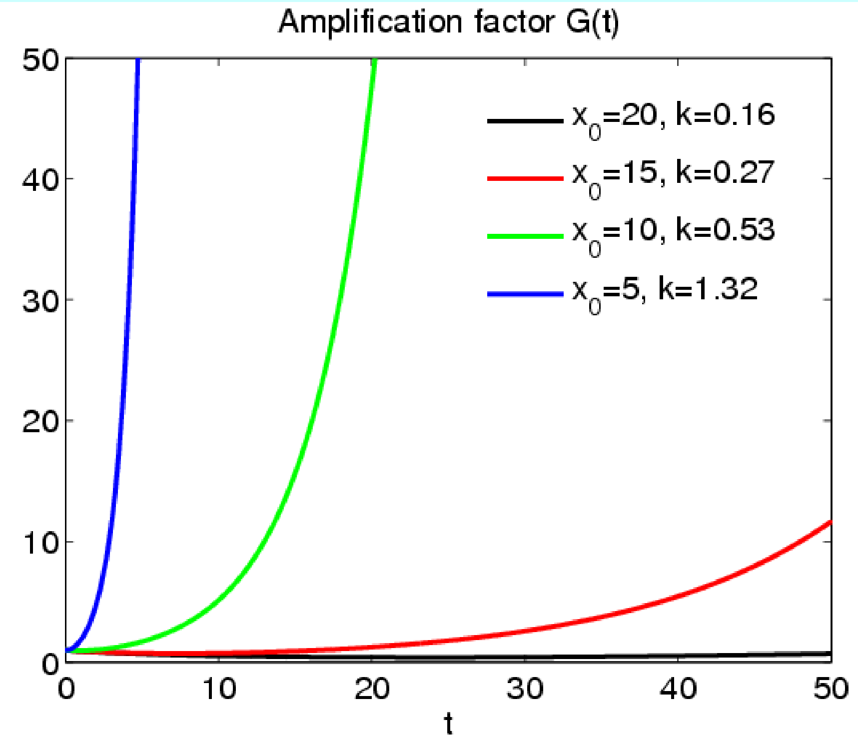


Amplification factor G(t), R=50, symmetric perturbations, $\beta_0 = 1$, k = 1.5.
(left): U(y); (right): U(x_0,y), x_0=10.

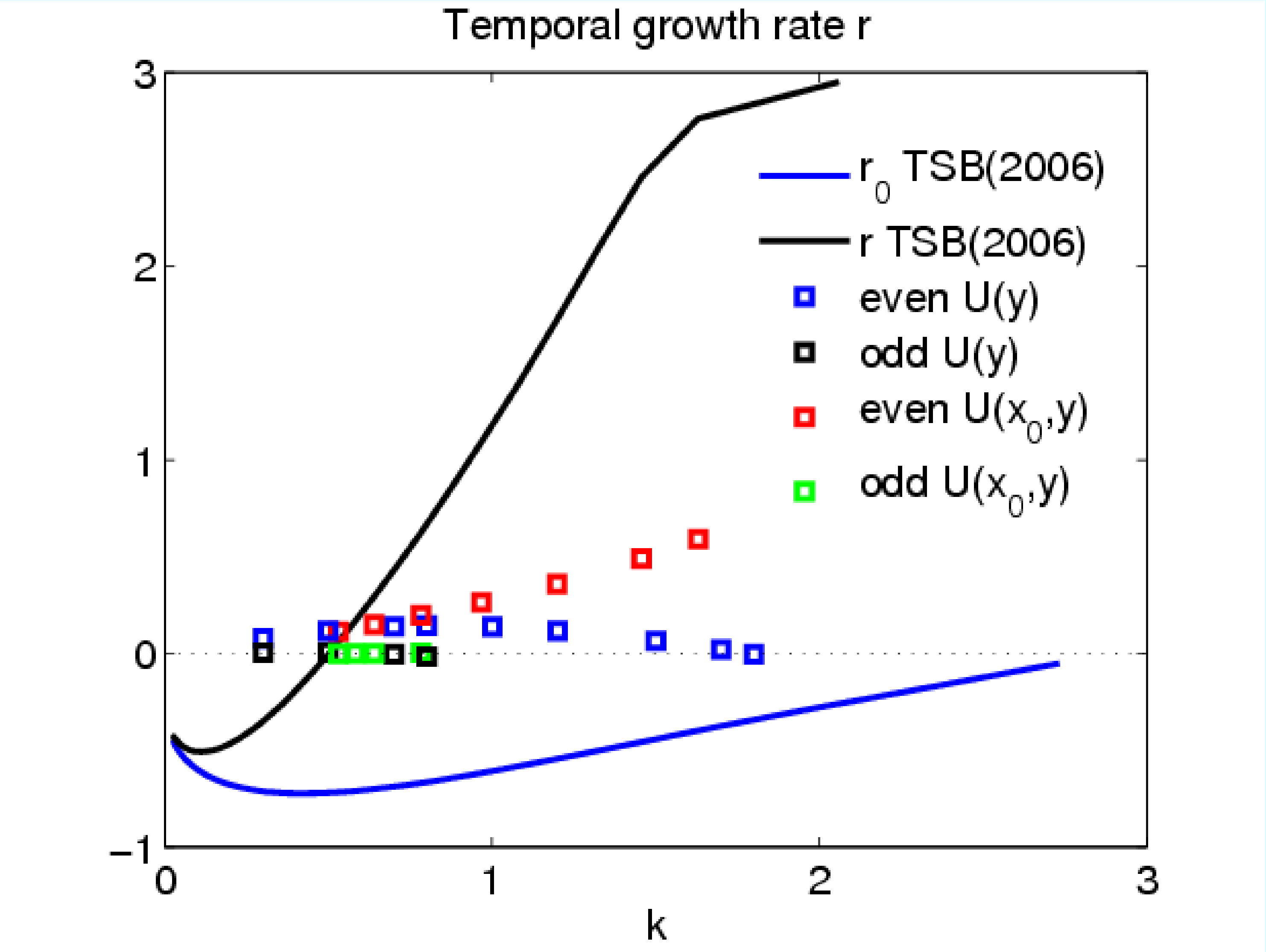


Amplification factor G(t), R=100, asymmetric perturbations, $\beta_o = 1$, $\Phi = \pi/2$.
(left): U(y); (right): $U(x_o,y)$, k=1.5.





Amplification factor G(t), R=100, symmetric perturbations, $\beta_0 = 1$, $\Phi = 0$. (left): U(y); (right): $U(x_0,y)$, x_0=20, 15, 10, 5 and k is the most unstable wavenumber (dominant saddle point) for every x_0 according to the dispersion relation in Tordella, Scarsoglio & Belan, 2006 PoF and Belan & Tordella, 2006 JFM



Temporal growth rate *r*, R=100, β_0 = 1, Φ = 0. Comparison between present results U(y) (black and blue squares) and $U(x_0,y)$ (red and green squares, where k is the most unstable wavenumber for every x_0) and Tordella, Scarsoglio & Belan, 2006 PoF, Belan & Tordella, 2006 JFM (solid lines)

Conclusions and incoming developments

- The linearized perturbation analysis considers both the early transient as well as the asymptotic behavior of the disturbance
- Three-dimensional (symmetrical and asymmetrical) initial disturbances imposed
- Numerical resolution of the resulting partial differential equations for different configurations
- Comparison with results obtained solving the Orr-Sommerfeld eigenvalue problem
- *More accurate description of the base flow (from a family of wakes profiles to a weakly non-parallel flow)*
- *Comparison with the inviscid theory*
- *Introduction of multiple spatial and temporal scales*
- *Optimization of the initial disturbances*