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Conference on System Modeling and Optimization

Analysis of the convective instability of the two-dimensional wake

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Introduction

A linear stability study is here presented for two dimensional non-parallel flows in the intermediate and far wake behind a circular body.

The hydrodynamic stability analysis is developed within the linear theory of normal modes; through a perturbative approach, it is observed the behavior of small oscillations applied to the base flow.

An analytic expression of the base flow according to Navier-Stokes model is given by an asymptotic expansion (Tordella and Belan, 2003; Belan and Tordella, 2002), which considers non-parallelism effects (such as exchange of transverse momentum and entrainment).

It is supposed that the system slowly evolves in space (Tordella and Belan, 2005) and also in time; using multiple spatial and temporal scales, we can verify how this evolution influences the stability characteristics and discuss about a validity domain for parallel flow.

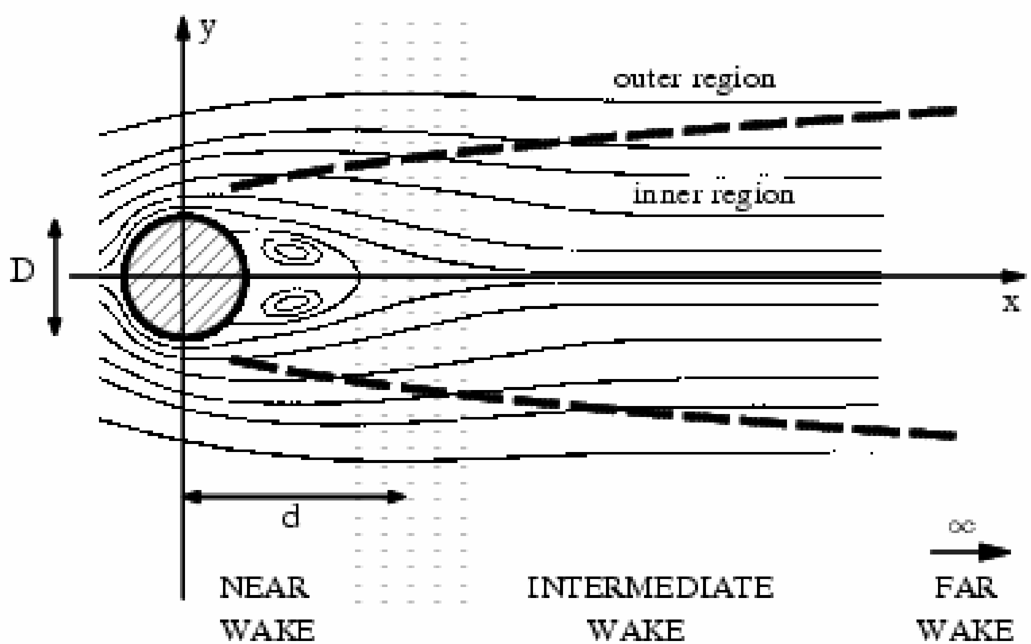
Basic equations and physical problem

Steady, incompressible and viscous base flow described by continuity and Navier-Stokes equations with dimensionless quantities $U(x,y)$, $V(x,y)$, $P(x,y)$ and $\rho \approx$cost

$$\begin{cases} \partial_x U + \partial_y V = 0 \\ U\partial_x U + V\partial_y U + \frac{1}{\rho}\partial_x P = \frac{1}{R}\nabla^2 U \\ U\partial_x V + V\partial_y V + \frac{1}{\rho}\partial_y P = \frac{1}{R}\nabla^2 V \end{cases}$$

$R = \rho U_c D/\mu$

Boundary conditions: symmetry to x, uniformity at infinity and field information in the intermediate wake



To analytically define base flow, its domain is divided into two regions both described by Navier-Stokes model

Inner region flow -> $f_i = f_{i0}(\eta) + x^{-1/2} f_{i1}(\eta) + x^{-1} f_{i2}(\eta) + \cdots$,

$$\eta = y x^{-1/2}$$

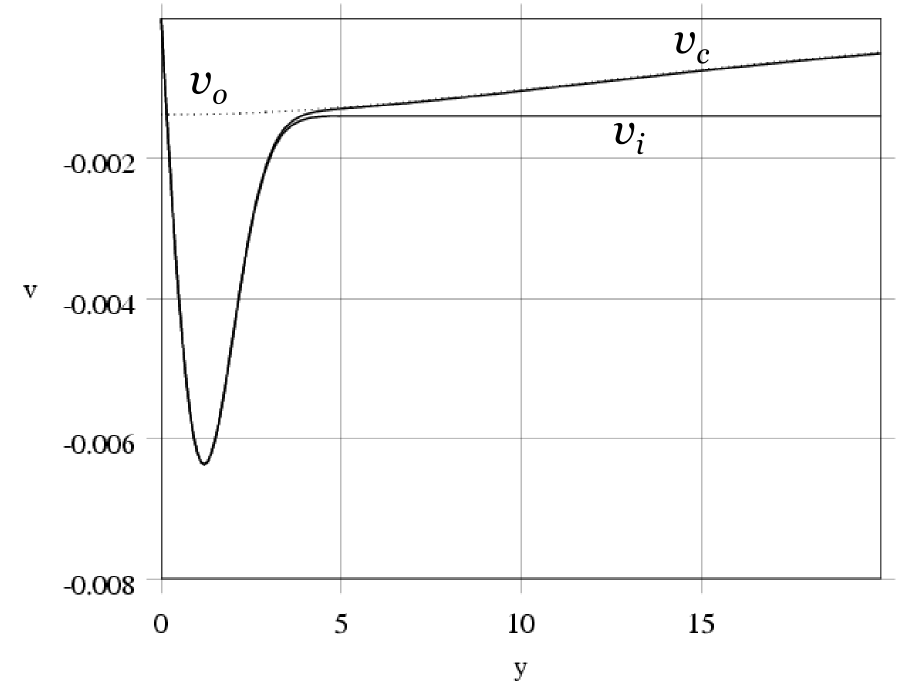
Outer region flow -> $f_o = f_{o0}(s) + r^{-1/2} f_{o1}(s) + r^{-1} f_{o2}(s) + \cdots$,

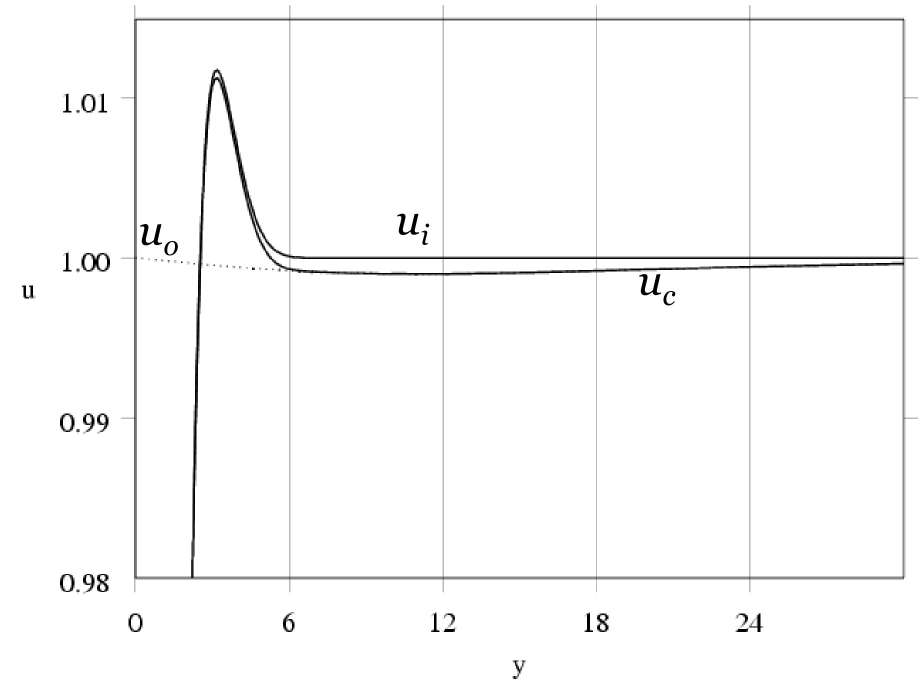
$$r = \sqrt{x^2 + y^2}, \; s = y x^{-1}$$

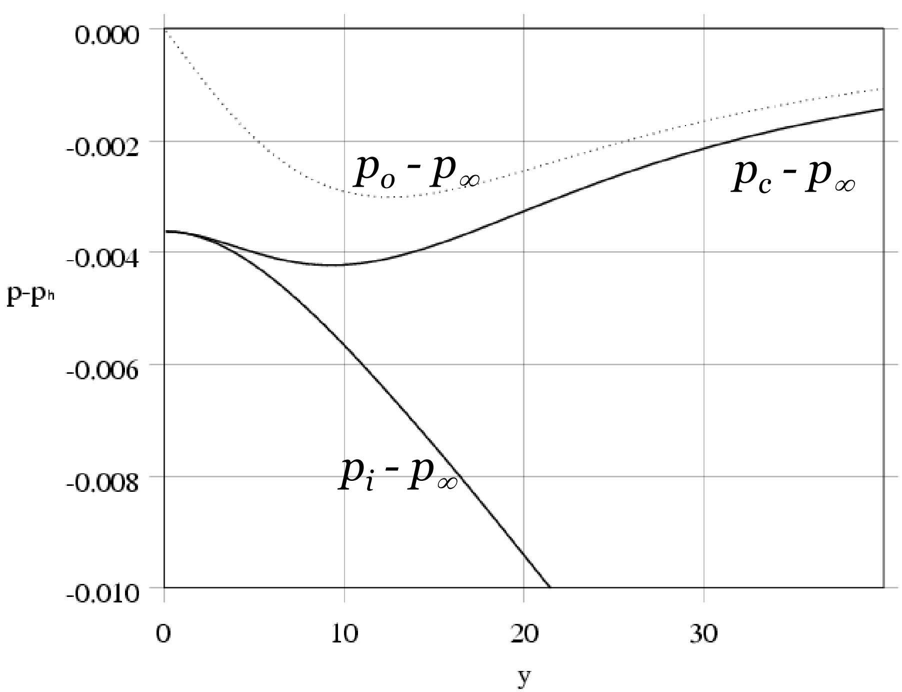
Physical quantities involved in matching criteria are the pressure longitudinal gradient, the vorticity and transverse velocity. Inner and outer expansions are used to obtain the composite expansion $f_{cn} = f_{in} + f_{on} - (f_{on})^{in}$ which is, by construction, continuous and differentiable over the whole domain.

Accurate representation of the velocity and pressure distributions (obtained without restrictive hypothesis) and analytical simplicity of expansions.

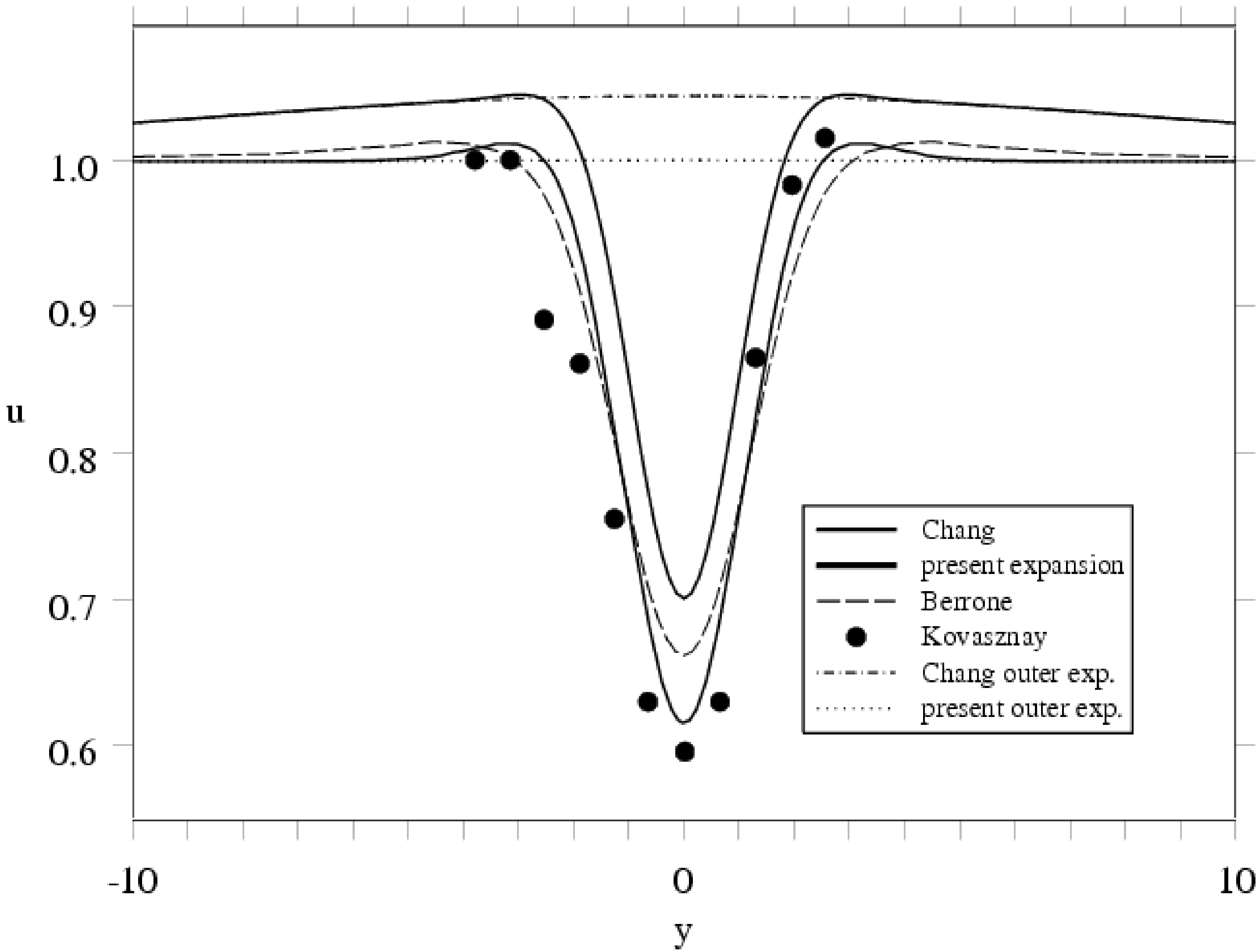
Here we take the inner expansion up to third order as base flow solution for the wake.







R = 34, x/D = 20. Fourth order of accuracy – Inner, outer and composite expansions for velocity and pressure.



R = 34, x/D = 20. Comparison of the present fourth order outer and composite, Chang's outer and composite (1961), Kovasznay's experimental (1948) and Berrone's numerical (2001) longitudinal velocity distributions.

Stability theory

Base flow is excited with small oscillations.

Perturbed system is described by Navier-Stokes model
$$\begin{cases} u^*(x,y,t) = U(x,y) + u(x,y,t) \\ v^*(x,y,t) = V(x,y) + v(x,y,t) \\ p^*(x,y,t) = P_o + p(x,y,t) \end{cases}$$

Subtracting base flow equations from those concerning perturbed flow and neglecting non linear oscillating terms, the linearized perturbative equation in term of stream function $\psi(x,y,t)$ is

$$\begin{cases} \partial_t \nabla^2 \psi + (\partial_x \nabla^2 \Psi)\psi_y + \Psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \Psi)\psi_x - \Psi_x \partial_y \nabla^2 \psi = \frac{1}{R}\nabla^4 \psi \\ \lim_{|y|\to\infty} \psi(x,y,t) = 0 \\ \lim_{|y|\to\infty} \partial_y \psi(x,y,t) = 0 \end{cases}$$

Normal modes theory $\longrightarrow$ $\psi(x,y,t) = Re(\varphi(x,y,t)\, e^{i\vartheta(x,t)})$

Perturbation is considered as sum of normal modes, which can be treated separately since the system is linear.

$\varphi(x,y,t)$ complex eigenfunction, $\vartheta(x,t) = h_0 x - \sigma_0 t$, $(\partial_x \vartheta = h_0,\ \partial_t \vartheta = -\sigma_0)$

$h_o = k_o + i\, s_o$ → complex wave number
$\sigma_o = \omega_o + i\, r_o$ → complex frequency

k_o: **wave number**
s_o: **spatial growth rate**
ω_o: **frequency**
r_o: **temporal growth rate**

Perturbation amplitude is proportional to $e^{-s_0 x + r_0 t}$

- $r_o > 0$ for at least one mode ➡ unstable flow
- $r_o \leq 0$ for all modes ➡ stable flow
- $s_o < 0$ for at least one mode ➡ convectively unstable flow
- $s_o \geq 0$ for all modes ➡ convectively stable flow

Convective instability: $r_o < 0$ for all modes, $s_o < 0$ for at least one mode. Perturbation spatially amplified in a system moving with phase velocity of the wave but exponentially damped in time at fixed point.

***Absolute* instability:** $r_o > 0$ ($v_g = \partial\omega_o/\partial k_o = 0$ local energy increase) for at least one mode. Temporal amplification of the oscillation at fixed point.

Stability analysis through multiscale approach

Slow spatial and temporal evolution of the system ⟶ slow variables $x_1 = \varepsilon x$, $t_1 = \varepsilon x$.

$\varepsilon = 1/R$ is a dimensionless parameter that characterizes non-parallelism of base flow.

Hypothesis: $\psi(x,y,t)$ and $\Psi(x,y,t)$ are expansions in term of ε:

$$\psi = [\varphi_0(x_1,y,t_1) + \varepsilon\varphi_1(x_1,y,t_1) + \cdots]\, e^{i\theta(x_1,t_1;\varepsilon)}$$

$$\partial_y\Psi = U_0(x_1,y) + \epsilon U_1(x_1,y) + \cdots \qquad \partial_x\Psi = -\epsilon V_1(x_1,y) + \cdots$$

By substituting in the linearized perturbative equation, one has

(ODE dependent on φ_0) + ε(ODE dependent on φ_0, φ_1) + $O(\varepsilon^2)$

Order zero theory. Homogeneous Orr-Sommerfeld equation (parametric in x_1).

$$\begin{cases} \mathcal{A}\zeta_0 = \sigma_0\mathcal{B}\zeta_0 \\ \zeta_0 \to 0,\ |y| \to \infty \\ \partial_y\zeta_0 \to 0,\ |y| \to \infty \end{cases}$$

$$\mathcal{A} = (\partial_y^2 - h_0^2)^2 - ih_0R[U_0(\partial_y^2 - h_0^2) - \partial_y^2U_0]$$

$$\mathcal{B} = -iR(\partial_y^2 - h_0^2)$$

where $\varphi_0(x_1,t_1,y) = A(x_1,t_1)\zeta_0(x_1,y)$, and $A(x_1,t_1)$ is the slow spatio-temporal modulation, determined at next order.

By numerical solution ⟶ eigenfunctions ζ_o and a discrete set of eigenvalues σ_{on}

First order theory. Non homogeneous Orr-Sommerfeld equation (x_1 parameter).

$$\begin{cases} \mathcal{A}\varphi_1 = \sigma_0 \mathcal{B}\varphi_1 + \mathcal{M}\varphi_0 \\ \varphi_1 \to 0, \ |y| \to \infty \\ \partial_y \varphi_1 \to 0, \ |y| \to \infty \end{cases}$$

$\mathcal{M}$ is related to base flow and consider non-parallel effects through transverse velocity presence

$$\begin{aligned} \mathcal{M} \ = \ & \{ \left[R(2h_0\sigma_0 - 3h_0^2 u_0 - \partial_y^2 u_0) + 4ih_0^3 \right] \partial_{x_1} \\ & + (Ru_0 - 4ih_0)\partial^3_{x_1 yy} - Rv_1(\partial_y^3 - h_0^2 \partial_y) + R\partial_y^2 v_1 \partial_y \\ & + ih_0 R \left[u_1(\partial_y^2 - h_0^2) - \partial_y^2 u_1 \right] + R(\partial_y^2 - h_0^2)\partial_{t_1} \} \end{aligned}$$

To obtain first order solution, the non homogeneous term is requested to be orthogonal to every solution of the homogeneous adjoint problem, so that

$$\begin{cases} \partial_t a(x,t) + K_2(x)\partial_x a(x,t) + \varepsilon K_1(x) = 0 \,,\, A(x,t) = e^{a(x,t)} \\ a(x, t=0) = c \\ \partial_x a(x \to \infty, t) = 0 \end{cases}$$

Keeping in mind that $\varphi_1(x_1,t_1,y) = A(x_1,t_1)\zeta_1(x_1,y)$, the complete problem gives

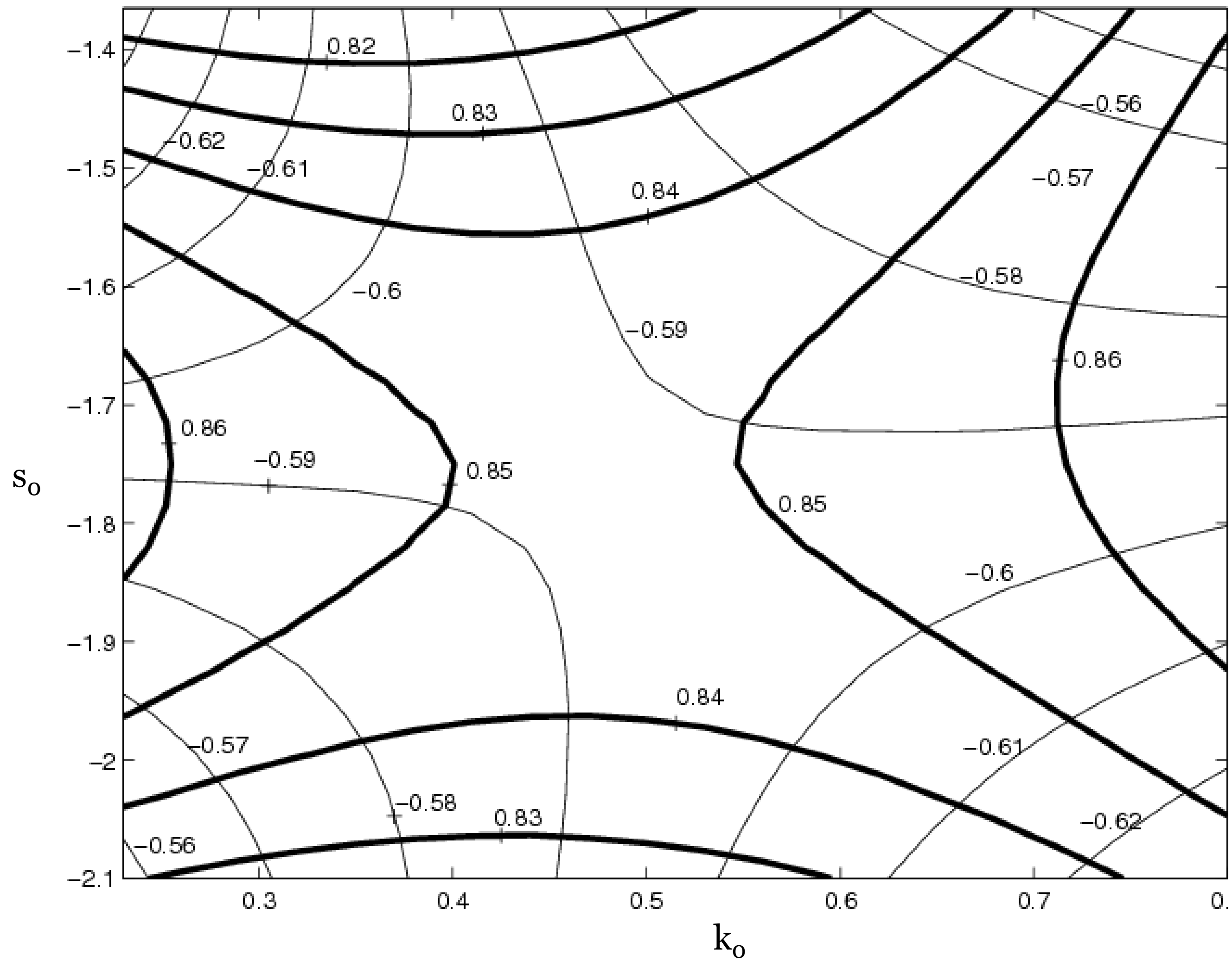
$$\psi = (\zeta_0 + \epsilon\zeta_1)e^{i\theta + a}$$

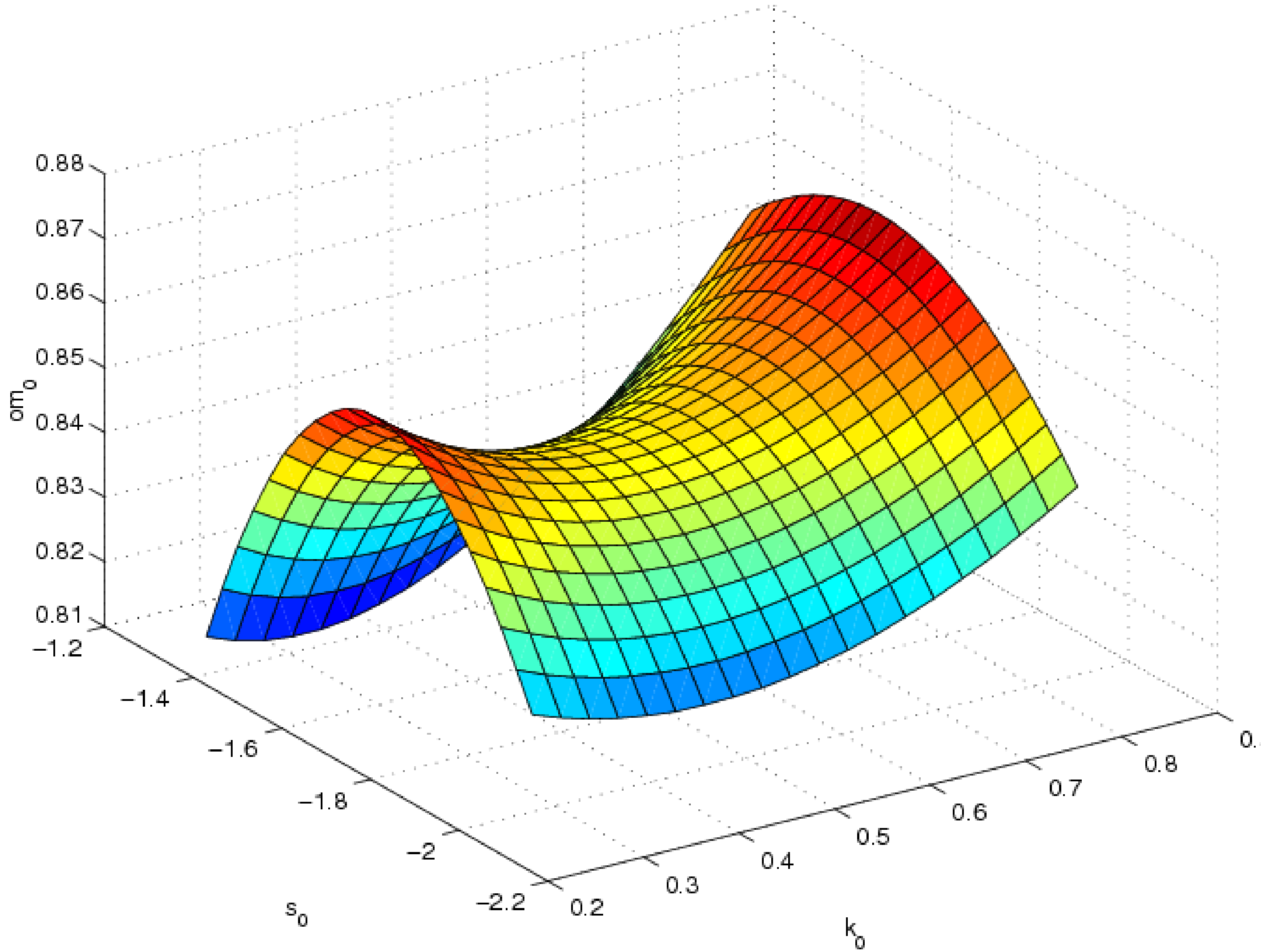
First order corrections h_1 e σ_1 are obtained by resolving numerically the evolution equation for modulation and differentiating numerically *a(x,t)* with respect to slow variables.

Perturbative hypothesis – Saddle points sequence

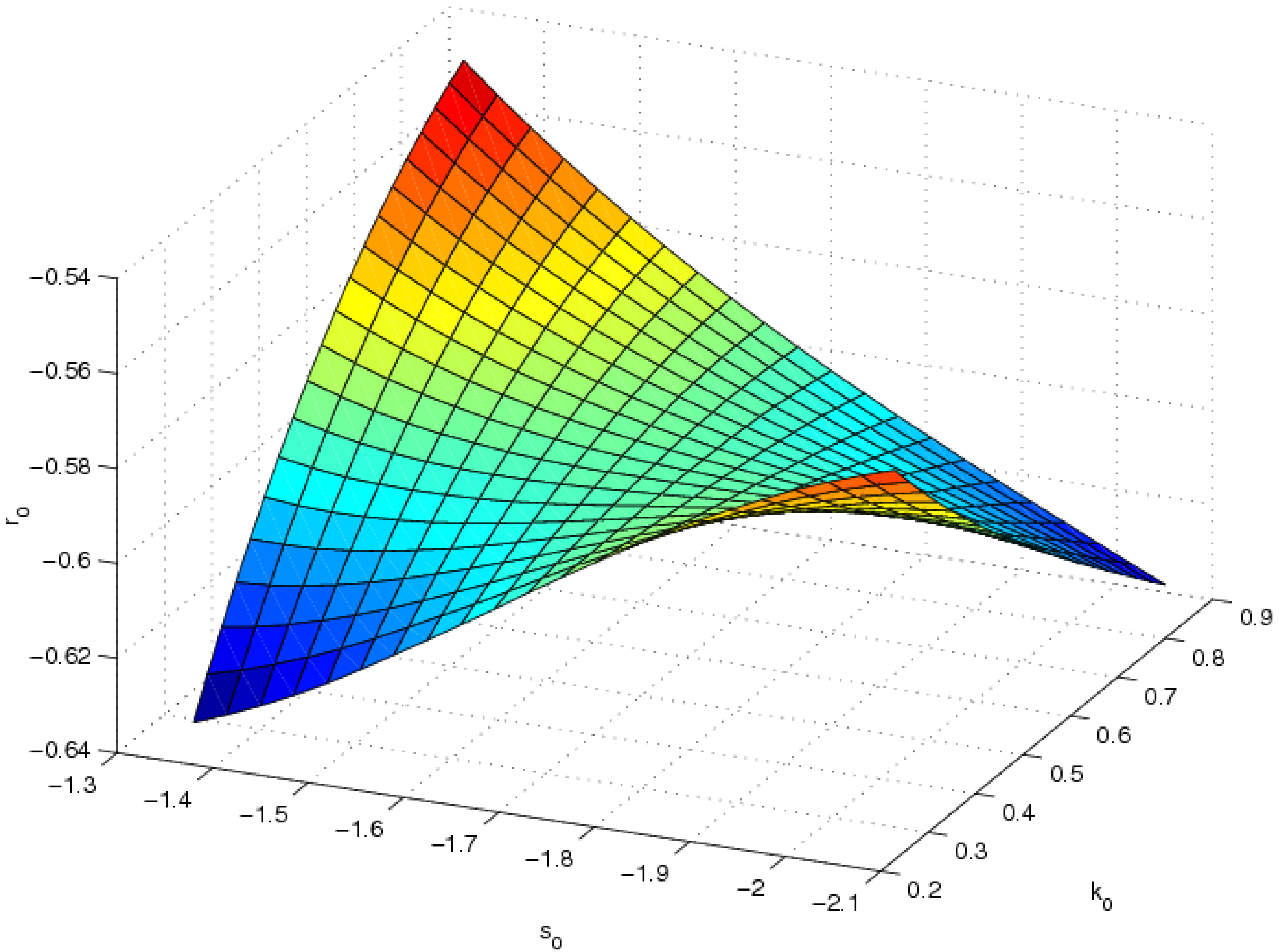
From order zero theory it's possible having a first approximation of the dispersion relation $\sigma_o=\sigma_o(h_o, x, R)$; for fixed values of x and R we individuate the saddle point (h_{os}, σ_{os}), that satisfies condition $\partial\sigma_o / \partial h_o = 0$, by selecting the eigenvalue with the largest imaginary part, using multidimensional maps

R = 35, x/D = 4. Frequency and temporal growth rate – Level curves. ω_o=cost (thick curves), r_o=cost (thin curves)





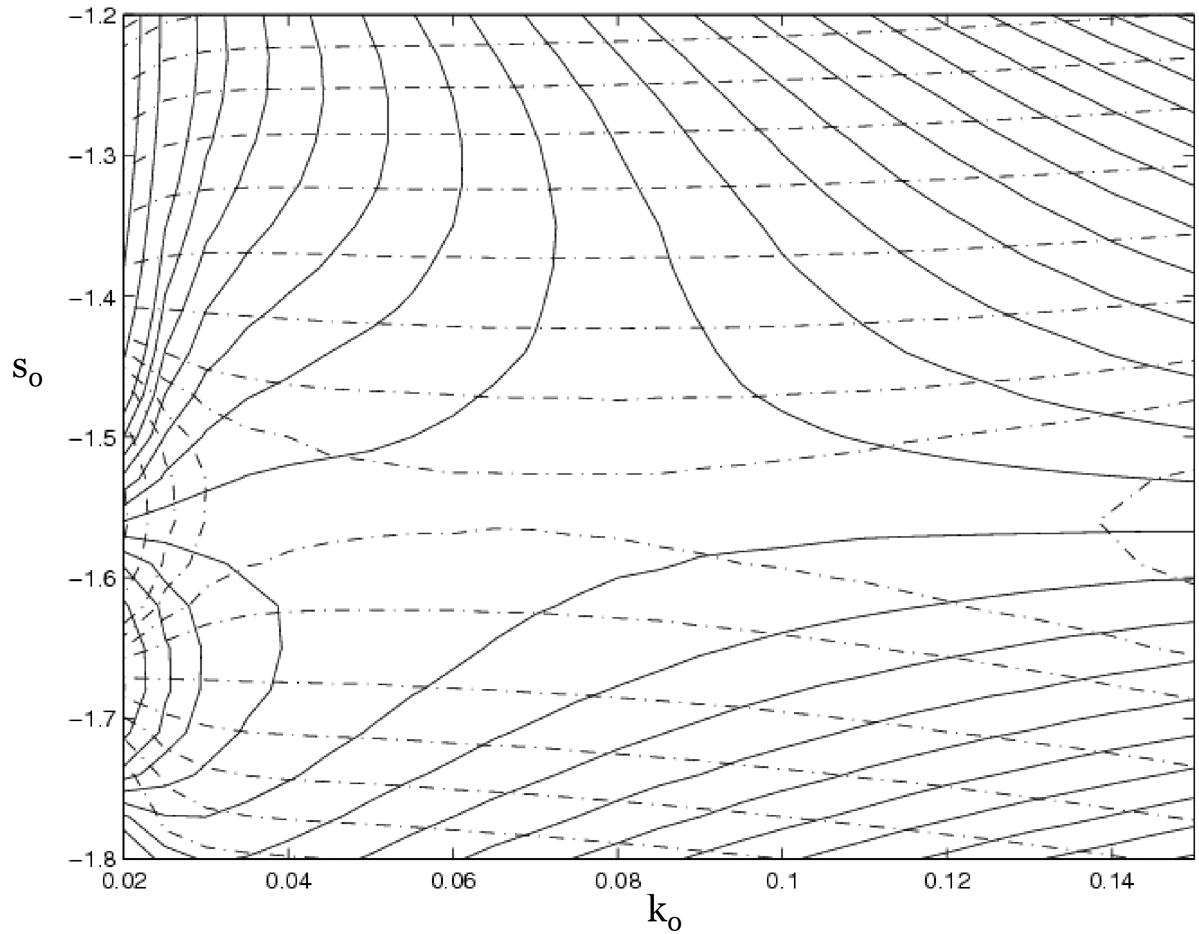
$\omega_o(k_o,s_o)$ - R = 35, x/D = 4.



$r_o(k_o,s_o)$ - R = 35, x/D = 4.

Saddle points determination is very sensitive to Orr-Sommerfeld boundary conditions and to number and choice of collocation points for order zero numerical resolution.

R = 50, x/D = 7. Frequency and temporal growth rate – Level curves. ω_o=cost (dashed curves), r_o=cost (solid curves).

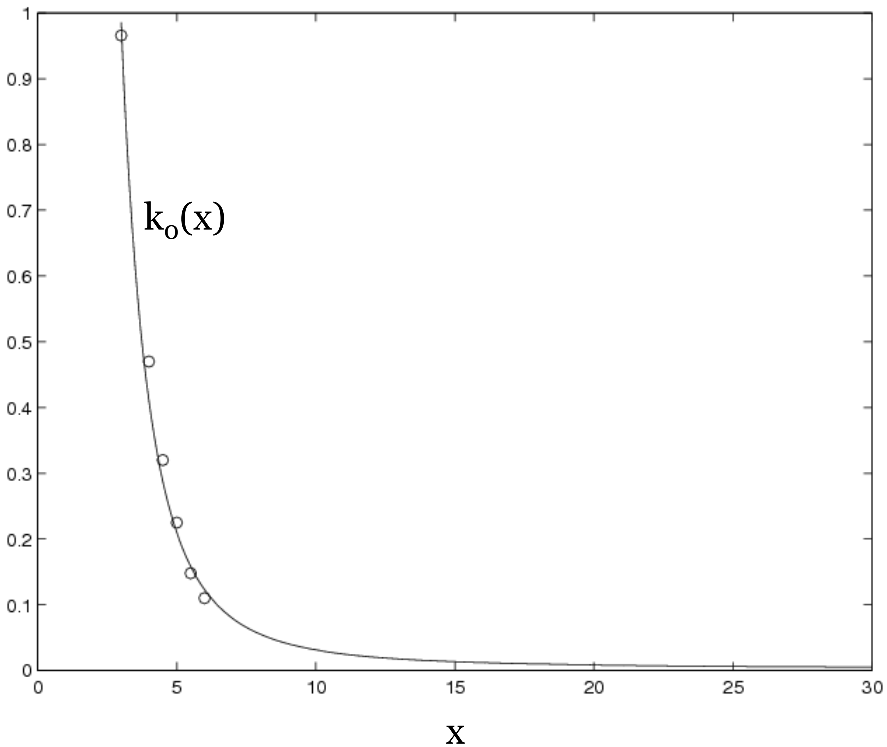


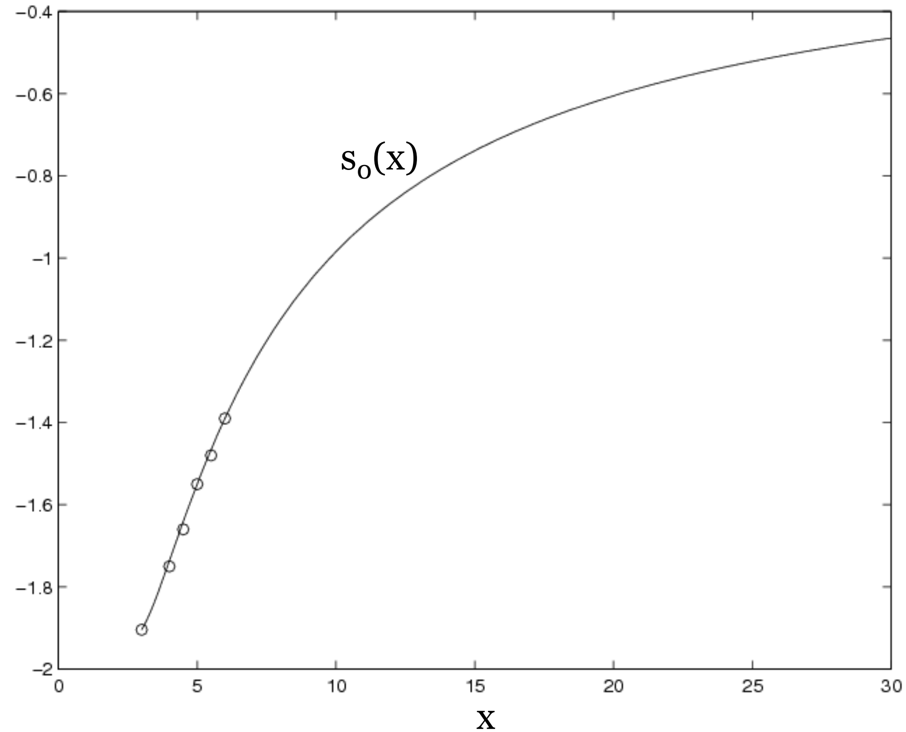
This aspect becomes more relevant when y-domain is getting larger, that is, for smaller R and larger x values.

For this reason, we use truncated Laurent series to extrapolate saddle points behavior in x from data at lower x values, that are more accurate.

$$k_0(x) = c_{k1}x^{-1} + c_{k2}x^{-2} + c_{k3}x^{-3}$$

$$s_0(x) = c_{s1} + c_{s2}x^{-1} + c_{s3}x^{-2}$$

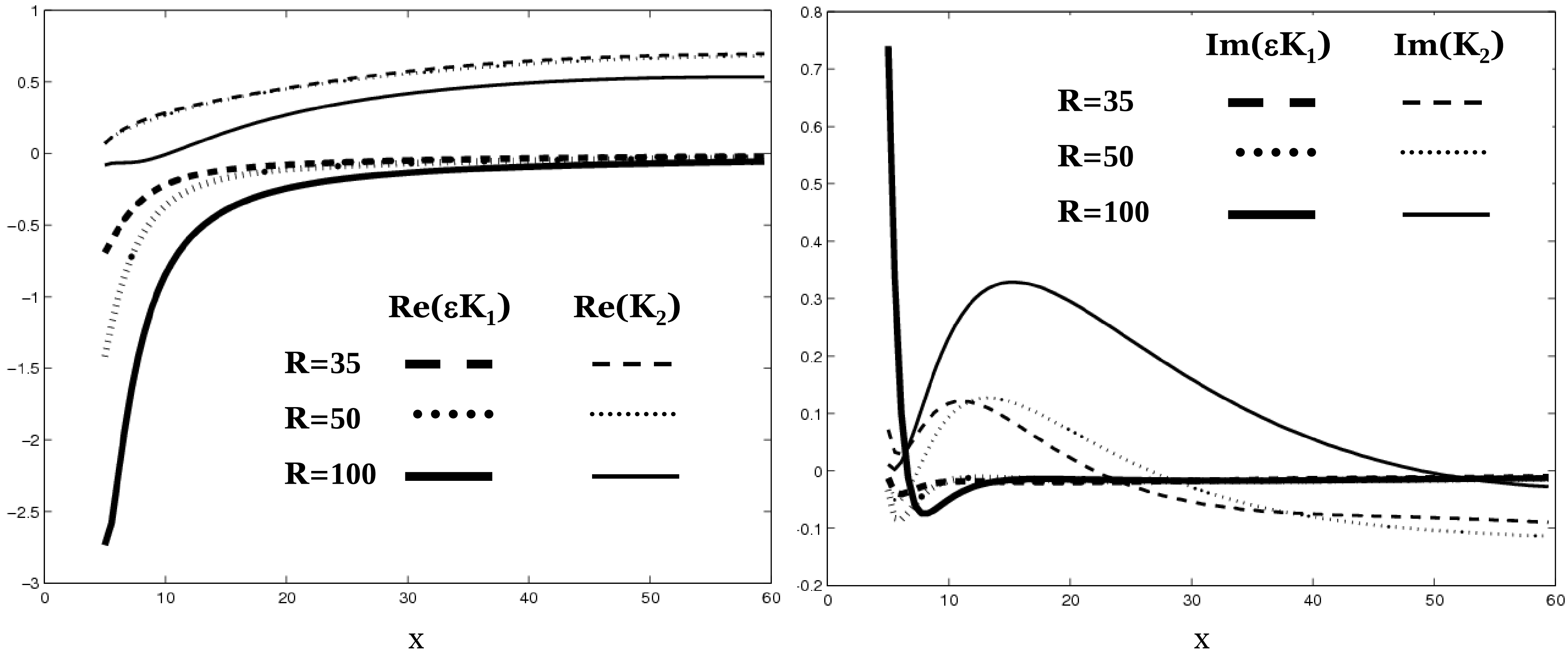




R = 35 – Saddle points (open circles) and extrapolated curve (solid line)

Once known $h_{os}(x)$ in this way, the relative $\sigma_{os}(x)$ are given by dispersion relation.

The system is now perturbed, at every longitudinal station, with those characteristics that at order zero turn out to be locally the most unstable (in absolute sense) for base flow.



Coefficients (Real and Imaginary part) of evolution equation for modulation

$\partial_t a(x,t) + K_2(x)\partial_x a(x,t) + \varepsilon K_1(x) = 0$ - R = 35, 50, 100

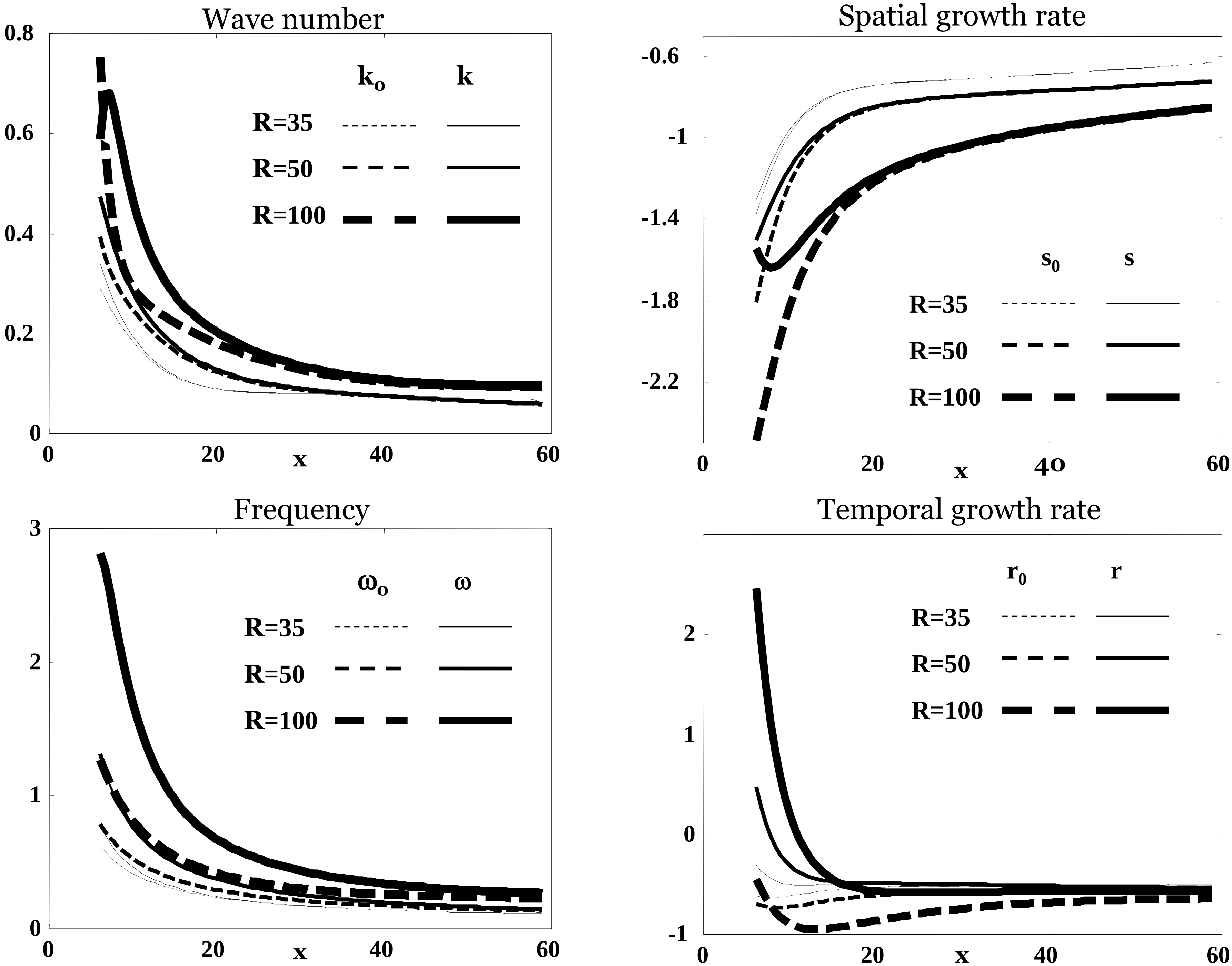
$$K_1(x) = \frac{J_1(x)}{J_3(x)}, K_2(x) = \frac{J_2(x)}{J_3(x)}$$

$$
\begin{aligned}
J_1(x_1) \;=\; & \int_{-\infty}^{\infty} \zeta_0^+ [(R(2h_0\sigma_0 - 3h_0^2 u_0 - \partial_y^2 u_0) + 4ih_0^3)\partial_{x_1} + \\
& +(Ru_0 - 4ih_0)\partial_{x_1yy}^3 - (ih_0 R(\partial_y^2 + h_0^2)u_1) + \\
& -(R(\partial_y^2 + h_0^2)v_1)\partial_y + (ih_0 Ru_1)\partial_y^2 + (Rv_1)\partial_y^3]\zeta_0 dy
\end{aligned}
$$

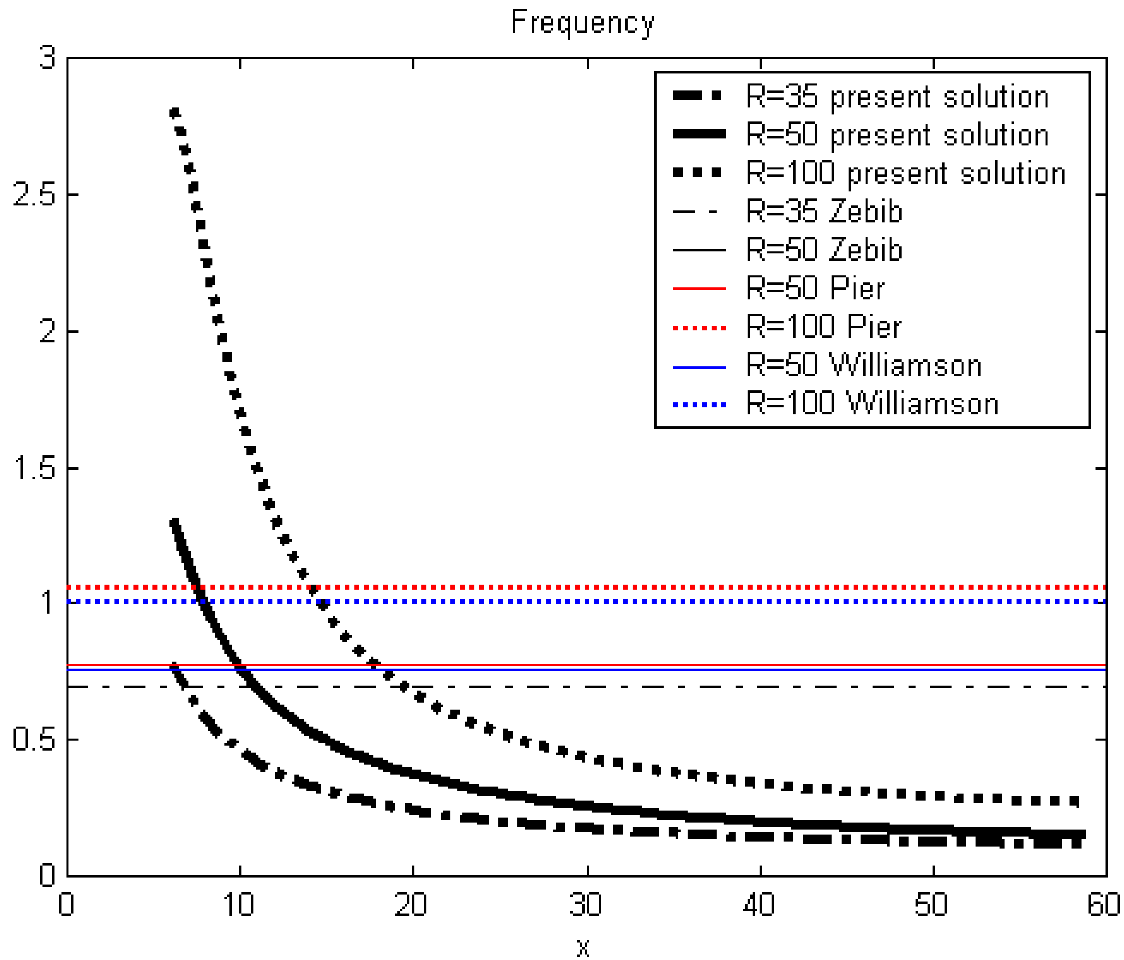
$$
\begin{aligned}
J_2(x_1) \;=\; & \int_{-\infty}^{\infty} \zeta_0^+ [(R(2h_0\sigma_0 - 3h_0^2 u_0 - \partial_y^2 u_0) + 4ih_0^3) + \\
& +(Ru_0 - 4ih_0)\partial_y^2]\zeta_0 dy
\end{aligned}
$$

$$
J_3(x_1) \;=\; \int_{-\infty}^{\infty} \zeta_0^+ \left[-Rh_0^2 + R\partial_y^2\right] \zeta_0 dy
$$

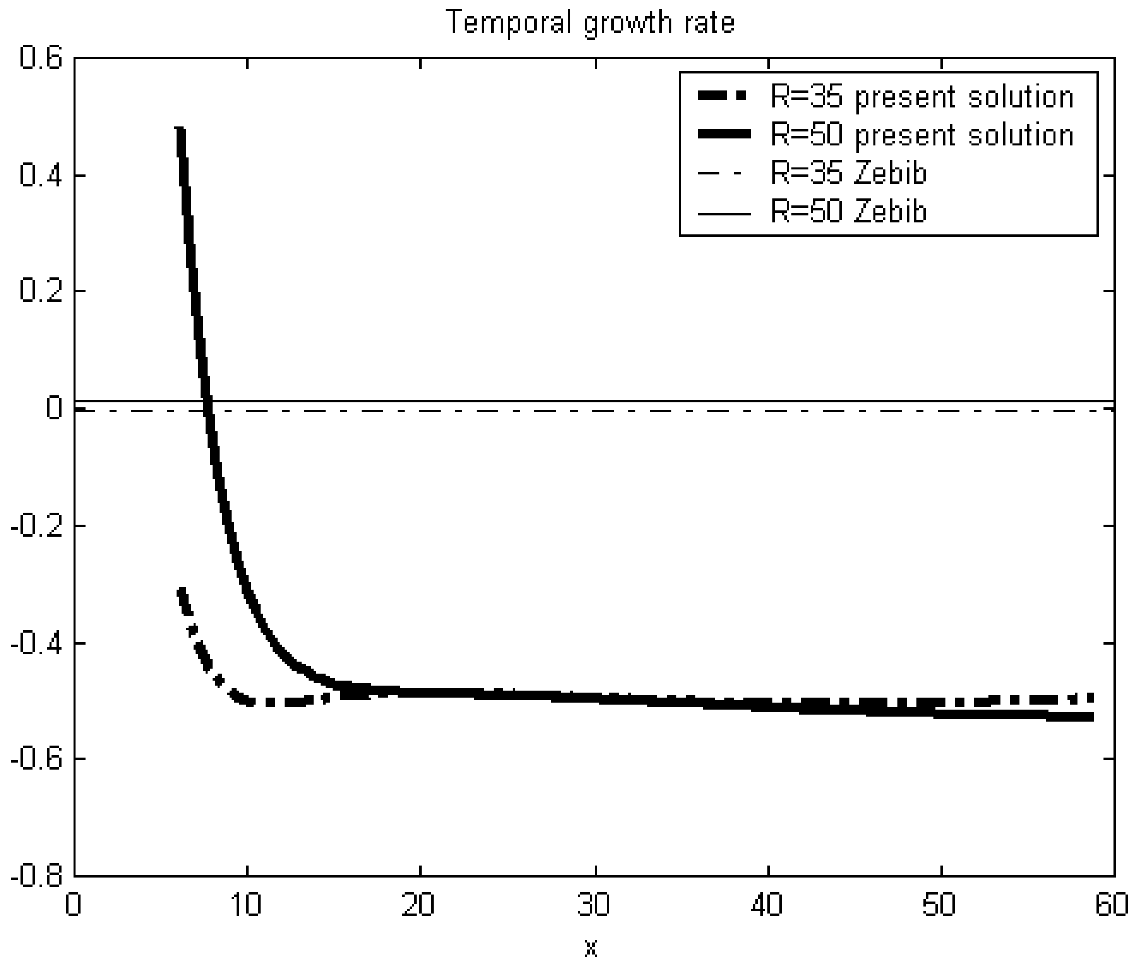
where ζ_0^+ is the adjoint eigenfunction



Wave number
0.8
0.6
0.4
0.2
0
0
20
40
60
x
k_o
k
R=35
R=50
R=100
Spatial growth rate
-0.6
-1
-1.4
-1.8
-2.2
0
20
40
60
x
s_0
s
R=35
R=50
R=100
Frequency
3
2
1
0
0
20
40
60
x
ω_o
ω
R=35
R=50
R=100
Temporal growth rate
2
1
0
-1
0
20
40
60
x
r_0
r
R=35
R=50
R=100



Frequency. Comparison between the present solution (R=35,50,100), Zebib's numerical study (1987), Pier's direct numerical simulations (2002), Williamson's experimental results (1988).



Temporal growth rate. Comparison between the present solution (R=35,50) and Zebib's numerical study (1987).

Validity limits for the near-parallel flow

First order corrections are acceptable when they are much lower than the corresponding order zero values; where they are not so, parallel flow theory is no longer valid.

A possible criterion to establish this, is the following

$$\left| \frac{1}{f(x)} \frac{\partial f(x)}{\partial x} \right| \ll \frac{1}{\lambda(x_*)}$$

where f is one of the stability characteristics and λ is the wave length in x^*.

For fixed R values, these conditions are more restrictive for temporal characteristics than for the spatial ones. Spatial growth rate s seems to be already well described at order zero, while frequency ω is the characteristic which is more influenced by first order corrections.

Increasing R, the region in which the flow can no more be considered parallel becomes larger; this region involves not only the near but also part of the intermediate wake.

Conclusions

***Validity limits for parallel theory*:** by observing first order corrections, the flow cannot be supposed parallel in the near wake and also in a relevant portion of the intermediate wake.

***System stability*:** for what said about acceptable first order corrections, the intermediate and far wake is convectively unstable. Positive temporal growth rate values are considered not acceptable, even if they are in a region of the domain (the beginning of near wake) in which they would be experimentally confirmed.

***Second order corrections (ε^2)*:** seem to be unnecessary, for they would not affect results so much in the region where parallel flow theory is valid and they would be completely useless where first order corrections are already too big.