# Pre-unstable set of multiple transient three-dimensional perturbation waves and the associated turbulent state in a shear flow

Daniela Tordella[1]
Stefania Scarsoglio[2]

[1]Department of Aeronautics and Space Engineering, Politecnico di Torino, Torino, Italy
[2]Department of Hydraulics, Politecnico di Torino, Torino, Italy

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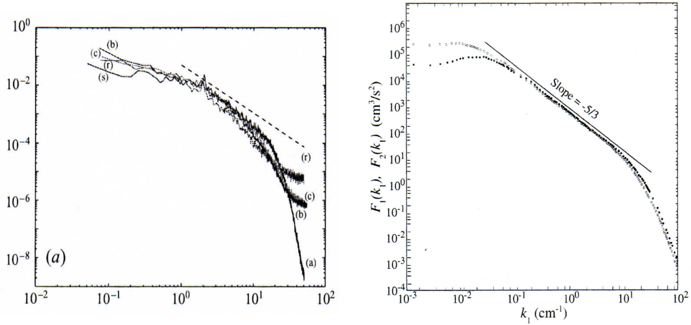


(left) *Evangelinos & Karniadakis, JFM 1999*. (right) *Champagne, JFM 1978*.

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    - Leaves aside the nonlinear interaction among the different scales;
- The perturbative evolution is ruled out by the **initial-value problem** associated to the Navier-Stokes linearized formulation.

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- Variety of the transient linear dynamics $\Rightarrow$ Understand how the energy spectrum behaves and compare it with the exponent of the developed turbulent state:
  - The difference is large $\Rightarrow$ quantitative measure of the nonlinear interaction in spectral terms;
  - The difference is small $\Rightarrow$ higher degree of universality on the value of the exponent of the inertial range, not necessarily associated to the nonlinear interaction.

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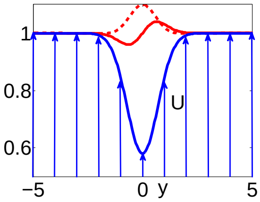


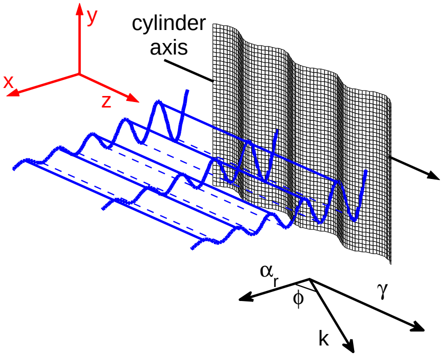

# Perturbative equations

- Perturbative linearized system:

$$\frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{v} = \hat{\Gamma}$$

$$\frac{\partial \hat{\Gamma}}{\partial t} = (i\alpha_r - \alpha_i)(\frac{d^2 U}{dy^2}\hat{v} - U\hat{\Gamma}) + \frac{1}{Re}[\frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\Gamma}]$$

$$\frac{\partial \hat{\omega}_y}{\partial t} = -(i\alpha_r - \alpha_i)U\hat{\omega}_y - i\gamma\frac{dU}{dy}\hat{v} + \frac{1}{Re}[\frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\omega}_y]$$

  The transversal velocity and vorticity components are $\hat{v}$ and $\hat{\omega}_y$ respectively, $\hat{\Gamma}$ is defined as $\widetilde{\Gamma} = \partial_x\widetilde{\omega}_z - \partial_z\widetilde{\omega}_x$.
- Initial conditions:
  - $\hat{\omega}_y(0, y) = 0$;
  - $\hat{v}(0, y) = e^{-y^2}\sin(y)$ or $\hat{v}(0, y) = e^{-y^2}\cos(y)$;
- Boundary conditions: $(\hat{u}, \hat{v}, \hat{w}) \to 0$ as $y \to \infty$.

# Perturbation energy

- Kinetic energy density *e*:

$$
\begin{aligned}
e(t;\alpha,\gamma) &= \int_{-y_d}^{+y_d} (|\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2) dy \\
&= \frac{1}{|\alpha^2+\gamma^2|} \int_{-y_d}^{+y_d} (|\frac{\partial \hat{v}}{\partial y}|^2 + |\alpha^2+\gamma^2||\hat{v}|^2 + |\hat{\omega}_y|^2) dy
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- Temporal growth rate $r$ (*Lasseigne et al., J. Fluid Mech., 1999*):

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r(t;\alpha,\gamma) = \frac{log|e(t;\alpha,\gamma)|}{2t}, \quad t > 0
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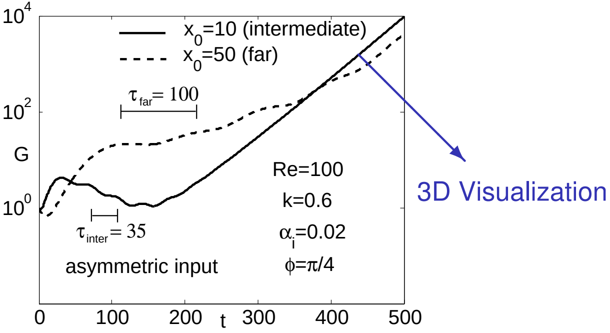

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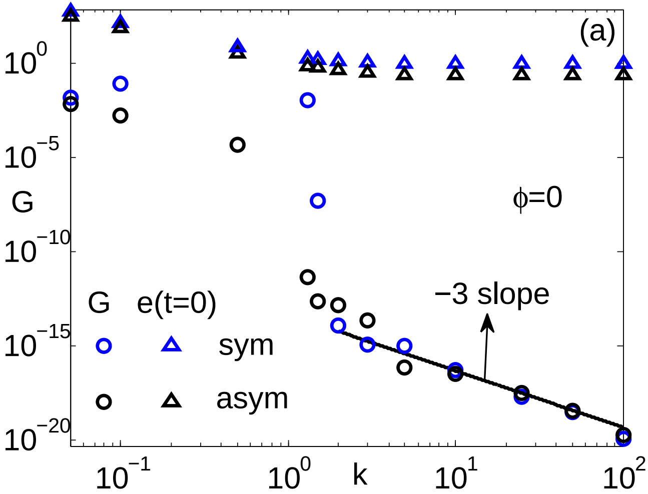
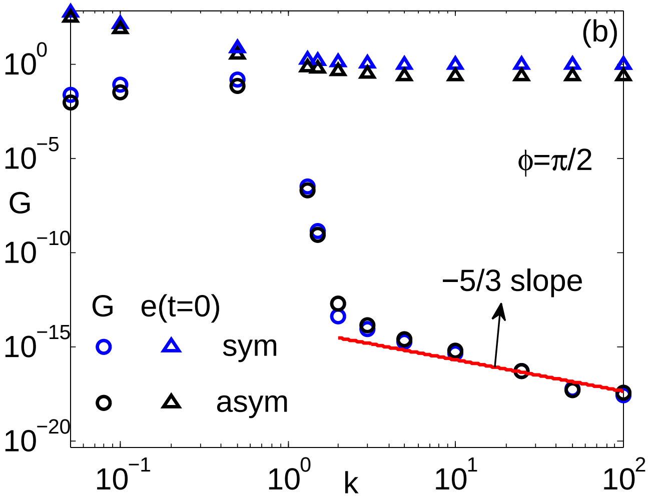
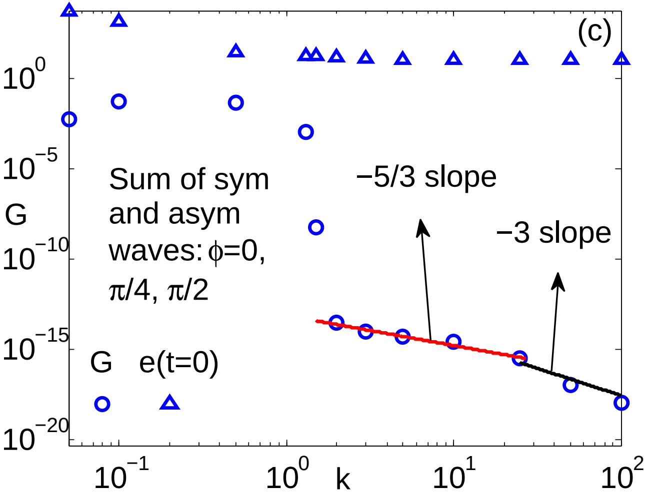


# Energy spectrum for transversal waves



# Energy spectrum of a 2D-3D combined perturbation




*Transient evolution of multiple three-dimensional waves.*

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*Coming next* $\Rightarrow$ Temporal observation window of a large number of small 3D perturbations injected in a statistical way into the system.