

Noise-induced spatial pattern formation

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Introduction

Spatial patterns are widely present in different natural dynamical systems. Their occurrence has been studied for quite a long time with several applications in different fields, from hydrodynamic systems (e.g. Rayleigh-Bénard convection), plant ecosystems (e.g. dryland and riparian vegetation), to biochemical and neural systems.

The study of patterns can offer useful information on the underlying processes causing possible changes of the system. Deterministic mechanisms in pattern formation have been widely studied [1] with a number of applications to environmental processes (see the review by Borgogno et al. 2009 [2]).

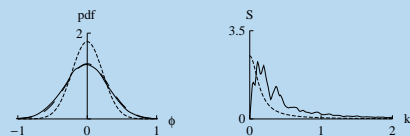
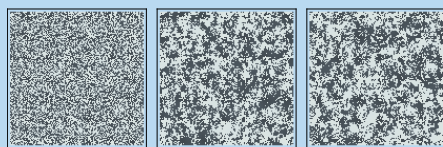
Stochastic models have only been developed more recently [3, 4]. They explain pattern formation as a noise-induced effect in the sense that patterns can emerge as a consequence of the randomness of the system's fluctuations.

Here, we propose an overview of the main stochastic processes related to the presence of a Gaussian white noise. In particular, we focus on the main components playing a crucial role in noise-induced pattern formation [5]: (i) a linear local dynamics, which damps the system to zero, (ii) an additive noise, which avoids the deterministic dynamics to decay, and (iii) a diffusive spatial coupling term, which provides spatial coherence.

We call **patterned** a field that exhibits an ordered state with organized spatial structures. This general definition, including both periodic as well as multiscale patterns, is often adopted in the environmental sciences, where the number of different processes can prevent the organization of the system with a clear dominant wavelength.

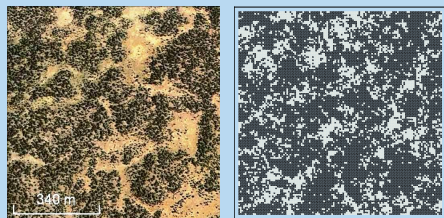
Additive Noise

$$\frac{\partial \phi}{\partial t} = -\phi + D\nabla^2 \phi + \xi$$



(top) Numerical simulation of the spatial field ϕ at $t = 0, 10, 100$, with $D = 50$, $s = 5$. (below) Pdf (solid: numerical simulation, dotted: classic mean-field analysis, dashed: corrected mean-field analysis) and azimuthal-averaged power spectrum S (solid: numerical simulations, dotted: structure function) of ϕ at $t = 100$.

$$\frac{\partial \phi}{\partial t} = -\phi + D\nabla^2 \phi + \xi + \mu$$



(left) Aerial photograph of vegetation pattern in New Mexico ($34^{\circ}47'N$, $108^{\circ}21'O$) and (right) numerical simulation at $t = 100$, $D = 80$, $s = 2$, $\mu = 0.1$. Google Earth imagery © Google Inc. Used with permission.

References

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Stochastic Modeling: General Framework

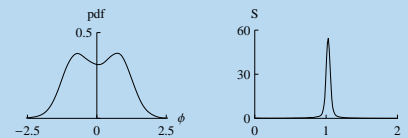
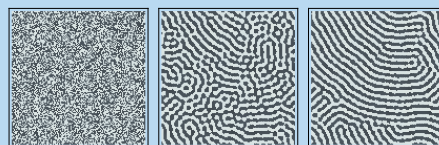
The spatio-temporal dynamics of the state variable, ϕ , can be expressed, at any point $\mathbf{r} = (x, y)$, as

$$\frac{\partial \phi}{\partial t} = f(\phi) + g(\phi)\xi(\mathbf{r}, t) + D\mathcal{L}[\phi] + h(\phi)F(t)$$

- $f(\phi)$: **local dynamics** (in the absence of spatial interactions with other points of the domain);
- $g(\phi)\xi$: **noise component**, where ξ is a zero-mean Gaussian white (in space and time) noise with intensity s ;
- $D\mathcal{L}[\phi]$: **spatial coupling**. \mathcal{L} represents the Laplacian (∇^2) or the Swift-Hohenberg ($\nabla^2 + k_0^2$)² coupling (k_0 is the selected wavenumber). D is the strength of the spatial coupling;
- $h(\phi)F(t)$: **time-dependent forcing term**, which can be in general modulated by a function, $h(\phi)$, of the local state of the system.

Multiplicative Noise

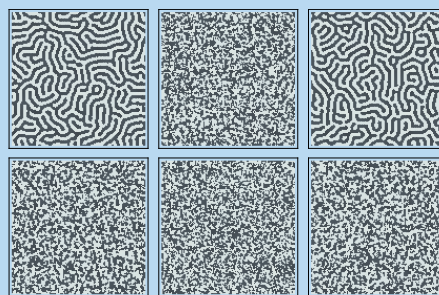
$$\frac{\partial \phi}{\partial t} = -\phi - \phi^3 + \phi\xi - D(\nabla^2 + k_0^2)\phi$$



(top) Numerical simulations of the spatial field ϕ at $t = 0, 10, 100$, $D = 15$, $s = 5$, $k_0 = 1$. (below) Pdf and azimuthal-averaged power spectrum S of ϕ at $t = 100$.

Stochastic Resonance

$$\frac{\partial \phi}{\partial t} = [-k + \alpha \sin(\omega t)]\phi - \phi^3 - D(k_0^2 + \nabla^2)\phi + \xi$$



Numerical simulations of the spatial field ϕ with $\alpha = k_0 = 1$, $k = 0.1$, $\omega/2\pi = 0.012$, and $D = 1$. The columns refer to 110, 140, 195 time units, the two rows correspond to $s = 2.5 \cdot 10^{-2}$ and $s = 2.5$, respectively.

Conclusions

Three main components play a fundamental role in the mechanism of noise-induced pattern formation: (i) a **deterministic local dynamics**, which tends to drive the field variable to a uniform steady state, (ii) an **additive noise** able to maintain the dynamics away from the uniform steady state, and (iii) a **spatial coupling term** which provides spatial coherence. For high enough multiplicative noise intensity, the spatial coupling exploits the initial instability giving rise to ordered structures. The presence of a temporal periodicity leads to oscillating patterns which periodically emerge and disappear. Since noisy fluctuations are always present in real systems and pattern formation, here described, is **completely noise-induced**, randomness can actually promote spatial coherence in different environmental processes.