Real rank boundaries and loci of forms

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Introduction

Main motivations:

- Classical projective geometry of loci of points.
- Connections between real algebraic geometry and tensor ranks.
- Suppose we have a description of the real rank boundary between two real ranks. Informally, if the equation of this real rank boundary almost vanishes on our tensor, it means that we are in the vicinity of it. Close to the boundary, usually numerical methods approximating rank are less accurate.

Ranks and X-ranks

Let $\mathbb{K}$ be a field (for us $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $X \subset \mathbb{P}^N_{\mathbb{K}}$ a projective variety not contained in a hyperplane.

Definition

The X-rank of a point $f \in \mathbb{P}^N_{\mathbb{K}}$, denoted $\mathrm{rk}_X(f)$, is the minimum integer s such that f is in the span of s distinct points of X:

$$f \in \langle \ell_1, \dots, \ell_s \rangle, \text{ where } \ell_i \in X.$$

- $X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$ (Veronese variety of degree d) $\rightsquigarrow$ Waring rank of homogeneous polynomials/**forms** (or symmetric rank of symmetric tensors);
- $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}$ (Segre variety) $\rightsquigarrow$ tensor rank of tensors in $\mathbb{K}^{n_1+1} \otimes \cdots \otimes \mathbb{K}^{n_d+1}$.

Typical ranks

Definition

Let $\mathbb{P}^N_{\mathbb{R}}$ be the real projective space equipped with the euclidean topology. Let $X \subset \mathbb{P}^N_{\mathbb{R}}$ be a real projective variety. The sets $\mathcal{R}_s = \{f \in \mathbb{P}^N_{\mathbb{R}} \mid \mathrm{rk}_X(f) = s\}$ are semi-algebraic. If $\mathcal{R}_s$ contains an open euclidean ball, then s is a ***typical rank****.*

There might be several typical ranks (whereas over an algebraically closed field we have a unique typical rank, called **generic**).

For instance, consider $X = \nu_d(\mathbb{P}^1)$, case of binary forms of degree d.

Theorem (Blekherman, after Comon-Ottaviani)

Typical Waring ranks of binary forms of degree d take all integer values between $\lceil \frac{d+2}{2} \rceil$ and d.

More about typical ranks

Theorem (Blekherman-Teitler)

Let $X \subset \mathbb{P}^N_{\mathbb{R}}$ be a real projective variety. Let g be the complex generic rank of its complexification. Then g is the smallest typical rank of X.

Theorem (Bernardi-Blekherman-Ottaviani)

Let $X \subset \mathbb{P}^N_{\mathbb{R}}$ be a real projective variety. Then any X-rank between the lowest typical rank and the highest typical rank is also typical.

Problem

Classify typical ranks for special real algebraic varieties.

Ternary forms

For $X = \nu_d(\mathbb{P}^2)$, the case of ternary forms of degree d.

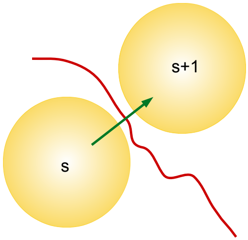
Theorem (Bernardi-Blekherman-Ottaviani)

- *Ternary cubics ($d = 3$) have only one typical rank, which is* 4.
- *Ternary quartics ($d = 4$) have at least two typical ranks (6 and 7), the maximum typical is 8 (open problem: is 8 realizable?)*
- *Ternary quintics ($d = 5$) have at least two typical ranks (7 and 8).*

Theorem (Michałek, Moon, Sturmfels, –)

Sextics ($d = 6$) have at least two typical ranks (10 and 11). Septics ($d = 7$) have at least two typical ranks (12 and 13).

Real rank boundaries



Definition

Let $X \subset \mathbb{P}^N_{\mathbb{R}}$ be a real projective variety. Let g = complex generic rank of its complexification in $\mathbb{P}^N$. Let

$$\mathcal{R}_X = \{f \in \mathbb{P}^N_{\mathbb{R}} \mid \mathrm{rk}_X(f) = g\}.$$

If X has more than one typical rank, $\partial\mathcal{R}_X$ is non-empty and codimension one. The **real rank boundary** of X = Zariski closure $\partial_{alg}\mathcal{R}_X$ over $\mathbb{C}$ (a hypersurface in $\mathbb{P}^N$).

Real rank boundaries for forms

For binary forms, a lot is known about the real rank boundary, along with all the boundaries dividing higher typical ranks, by Lee-Sturmfels and more recently by Brambilla-Staglianò.

For ternary forms, partially results are known for low degree forms. For instance:

Theorem (Michałek, Moon, Sturmfels, –)

The algebraic boundary for sextics is a hypersurface in the $\mathbb{P}^{27}$ *of ternary sextics. One of its irreducible components is the dual to the Severi variety of rational sextics.*

More:

Theorem (Michałek, Moon, Sturmfels, –)

The algebraic boundary for quintics is an irreducible hypersurface of degree 168 *in the* $\mathbb{P}^{20}$ *of ternary quintics.*

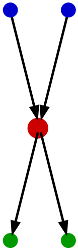
Yet another example: the hyperdeterminant of format $2 \times n \times n$

The next follows from theorems of Berqvist and De Silva–Lim.

Theorem

The real rank boundary between the only two typical ranks n and $n+1$ of $X = \mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ is the hypersurface defined by the hyperdeterminant.

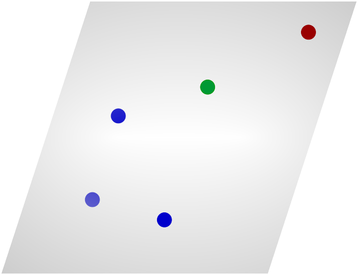
In this case, one can show $X^\vee = \tau(X) + \sigma_{n-2}(X)$: More generally, in several instances we expect one component of the real rank boundary to be a join of a tangential and a secant.



Carlini-Catalisano-Oneto forbidden loci

Definition

Let X be a projective variety in $\mathbb{P}^N$. The forbidden locus $\mathfrak{F}(f)$ of $f \in \mathbb{P}^N$ is the subset of points in X that **do not** *appear in any set $Z \subset X$ of cardinality $\mathrm{rk}_X(f)$ such that $f \in \langle Z \rangle$.*

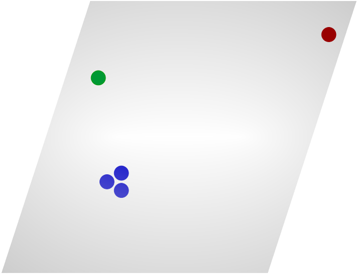


The blue and green points are in X, f is the red point, and $\mathrm{rk}_X(f) = 4$.
For instance, the green point **is not** in the forbidden locus.

Ranestad-Schreyer loci

Definition

Let X be a projective variety in $\mathbb{P}^N$. The Ranestad-Schreyer locus $\mathfrak{RS}(f)$, is the subset of points $\ell \in X$ with the following property: for every $\ell \in \mathfrak{RS}(f)$, there exists a zero-dimensional subscheme $Z \subset X$ of length $\mathrm{rk}_X(f)$ with $f \in \langle Z \rangle$, that has ℓ as non-reduced point.



The three (infinitesimally) close blue points represent a **double point** in the plane, because they are about to crash into one single point (the support). This double point has **non-reduced** structure and hence its support is in the Ranestad-Schreyer locus.

Example: Ternary cubics

The introduction of this locus was inspired by the following.

Theorem (Carlini-Catalisano-Oneto)

Let f be a general ternary cubic form. The forbidden locus $\mathfrak{F}(f)$ is closed. The two irreducible components are the Cayleyan and the dual of the Hessian of f.

This also appeared for determining an algebraic boundary sitting inside the variety of sums of powers $\mathrm{VSP}(f)$.

Proposition

$\mathfrak{R}\mathfrak{S}(f) = \mathfrak{F}(f)$.

Catalecticants and antipolars

Question: How to compute some Ranestad-Schreyer loci?

Let $X = \nu_{2d}(\mathbb{P}^n)$, Veronese variety of degree $2d$. It lives in the space of forms of degree $2d$ in $(n+1)$ variables: $\mathbb{P}^{\binom{2d+n}{n}-1}$. Let $S = \mathbb{C}[x_0, x_1, \ldots, x_n]$. We are interested in forms in S_{2d}.

By differentiation, $f \in S_{2d}$ induces a linear map: $C_f : S_d^* \to S_d$, called *middle catalecticant*.

Definition

Assume that C_f is an isomorphism. The antipolar of f is:

$$\Omega(f)(\ell) = \det(C_{f+\ell^{2d}}) - \det(C_f).$$

The form $\Omega(f)$ is defined in the coefficients of ℓ. Note: the forms $\Omega(f)$ and f have the same degree.

(This comes from:

$$v^t A^{-1} u = \frac{1}{\det(A)} \left[\det(A + uv^t) - \det(A)\right])$$

Example

Let $n = 2$ and $d = 2$. Let

$$f = 7/2x_0^4 + 9/7x_0^3x_1 + 1/2x_0^2x_1^2 + 2x_0x_1^3 + 5/8x_1^4 + 3x_0^3x_2 + 2x_0^2x_1x_2 + x_0x_1^2x_2 + 1/9x_1^3x_2 +$$

$$+ 2/9x_0^2x_2^2 + 6/7x_0x_1x_2^2 + 9/5x_1^2x_2^2 + 5/9x_0x_2^3 + 7/3x_1x_2^3 + 10/7x_2^4 \in \mathbb{C}[x_0, x_1, x_2]_4 = S_4.$$

Its middle catalecticant $C_f : S_2^* \to S_2$ has rank 6 (full-rank). Let $\ell = ax + by + cz$. Then (up to a constant)

$$\Omega(f)(\ell) = \det(C_{f+\ell^4}) - \det(C_f) =$$

$$= 1557939415221a^4 - 14648678086518a^3b + 165349467039743a^2b^2 - 296911950491100ab^3 + 26305857192150b^4 - 239845369655052a^3c +$$
$$295737285544470a^2bc - 850500178347330ab^2c + 240936112061400b^3c + 503631251967753a^2c^2 + 166673807209980abc^2$$
$$-957370784953905b^2c^2 - 9962940180690ac^3 + 710264007973800bc^3 - 193910604676425c^4,$$

so $\Omega(f)(\ell) \in \mathbb{C}[a, b, c]_4$.

Ranestad-Schreyer loci from antipolar

Theorem (Michałek-Moon, –)

Let $d \in \{1, 2, 3, 4\}$ and $f \in S_{2d}$ be a general ternary form of degree $2d$. Then $\mathfrak{RS}(f) = \{\Omega(f)(\ell) = 0\}$.

This works more generally for **toric varieties**:

- $X =$ projective toric variety.
- $S = \mathrm{Cox}(X)$ (in toric case it is a polynomial ring attached to X), graded by an abelian group called $\mathrm{Pic}(X)$.
  We can define catalecticants and antipolars in the toric context, obtaining the same result as above.

This leads to:

Theorem

Let $B \in \mathrm{Pic}(X)$ be a very ample line bundle. Let $f \in S_{2B}$ be a general form such that its X-rank coincides with the size of the catalecticant $S_B^* \to S_B$. The Ranestad-Schreyer locus $\mathfrak{RS}(f) = \{\Omega(f)(\ell) = 0\}$ is contained in the forbidden locus $\mathfrak{F}(f)$.

$\mathbb{P}^1 \times \mathbb{P}^1$ and its embeddings

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, $S = \mathbb{C}[x, y, z, w]$ **bigraded** (by $\mathbb{Z}^2$) with $\deg(x) = \deg(y) = (1, 0)$ and $\deg(z) = \deg(w) = (0, 1)$.

For $A = (u, v) \in \mathbb{Z}^2$, S_A denotes the vector space of forms of **bidegree** A.

The **Segre-Veronese embedding** of degree A:

$$\nu_{(u,v)} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{uv+u+v} = \mathbb{P}(S_A),$$

$$(\ell_1, \ell_2) \mapsto \ell_1^u \ell_2^v.$$

Example

For $A = (1, 1)$, we have the usual Segre embedding:

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3,$$

so that $\mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric surface in $\mathbb{P}^3$.

Real varieties of minimal degree

Varieties of minimal degree are projective varieties X such that their degree satisfies $\deg(X) = \operatorname{codim}(X) + 1$. They were classified by Bertini and Del Pezzo.

The following characterization is the key to (partially) describe the real rank boundary of a family of embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem (**Blekherman-Smith-Velasco**)

Let $X \subset \mathbb{P}^N_{\mathbb{R}}$ be a real irreducible non-degenerate projective variety such that the set of real points is Zariski dense. Every non-negative real quadratic form on $X(\mathbb{R})$ is a sum of squares of linear forms if and only if X is a variety of minimal degree.

Real rank boundaries for embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$

The Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ of degrees $A = (2, 2d)$ are varieties (surfaces) of minimal degree, examples of rational normal scrolls.

Theorem

For every $d \geq 1$, the real rank boundary of the real variety $\mathbb{P}^1 \times \mathbb{P}^1$ embedded with $A = (2, 2d)$ is non-empty. One of its components is the discriminant of the antipolar $\Omega(f)$, where $f \in S_A$.

This generalizes to $\mathbb{P}^n \times \mathbb{P}^1$ and $A = (2, 2d)$.

Thank you