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A refinement of the symmetric tensor decomposition algorithm

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Outline

[BCMT] Symmetric tensor decomposition, 2010

J. Brachat, P. Comon, B. Mourrain and E. Tsigaridas.
Linear Algebra and its Applications, 433 (11–12), pp. 1851-1872.

You have...

$$F = -4xy + 2xz + 2yz + z^2.$$

$$\Downarrow$$

You want...

$$F = (x-y)^2 - 2(x+y)^2 + (x+y+z)^2.$$

The problem
Decomposing symmetric tensors

You have...

$$F = -4xy + 2xz + 2yz + z^2,$$
$$f = F_{x=1} = -4y + 2z + 2yz + z^2.$$

$$\Downarrow$$

You want...

$$F = (x - y)^2 - 2(x + y)^2 + (x + y + z)^2.$$
$$f = F_{x=1} = (1 - y)^2 - 2(1 + y)^2 + (1 + y + z)^2.$$

$$R = \mathbb{K}[x_1, \ldots, x_n].$$

Apolar polynomial

$$f = \sum_{|\alpha| \leq d} f_\alpha \mathbf{x}^\alpha \in R_{\leq d}$$

$$\downarrow$$

$$f^* : R_{\leq d} \to \mathbb{K},$$

$$g = \sum_{|\alpha| \leq d} g_\alpha \mathbf{x}^\alpha \mapsto \langle f, g \rangle = \sum_{|\alpha| \leq d} \frac{f_\alpha g_\alpha}{\binom{d}{\alpha}}$$

Ideas

Move and solve the problem in the dual space

Apolar polynomial

$$f^* = \Big(\sum_{|\alpha|\leq d} f_\alpha \mathbf{x}^\alpha\Big)^* : R_{\leq d} \to \mathbb{K},$$

$$g = \sum_{|\alpha|\leq d} g_\alpha \mathbf{x}^\alpha \mapsto \langle f, g\rangle = \sum_{|\alpha|\leq d} \frac{f_\alpha g_\alpha}{\binom{d}{\alpha}}$$

Dual map

$$R_{\leq d} \hookrightarrow R^*_{\leq d},$$

$$f = \sum_{i=1}^{r} \lambda_i (1 + l_{1i}x_1 + \cdots + l_{ni}x_n)^d \mapsto f^* = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{(l_{1i},\ldots,l_{ni})}.$$

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Aim

Find $\Lambda \in R^*$ that restricts to f^* on $R_{\leq d}$:

$$\Lambda|_{R_{\leq d}} = f^*.$$

Let $\Lambda \in R^*$. We define

- the Henkel operator of Λ as

$$\begin{aligned} H_\Lambda : R &\to R^*, \\ r &\mapsto r \star \Lambda = \big(t \mapsto \Lambda(rt)\big), \end{aligned}$$

- $I_\Lambda = \ker H_\Lambda$,
- $\mathcal{A}_\Lambda = R/I_\Lambda$,
- the multiplication by r operators on $\mathcal{A}_\Lambda$ and $\mathcal{A}_\Lambda^*$ as

$$\begin{aligned} M_r : \mathcal{A}_\Lambda &\to \mathcal{A}_\Lambda, \\ t &\mapsto r \cdot t, \end{aligned} \qquad \begin{aligned} M_r^t : \mathcal{A}_\Lambda^* &\to \mathcal{A}_\Lambda^*, \\ \phi &\mapsto r \star \phi. \end{aligned}$$

[BCMT] Theorem

Let $\Lambda \in R^*$ and $r \in \mathbb{N}_{>0}$. The following are equivalent:

- There exist non-zero constants $\{\lambda_i\}_{i\in\{1,\dots,r\}} \subseteq \mathbb{K} \setminus \{0\}$ and distinct points $\{\zeta_i\}_{i\in\{1,\dots,r\}} \subseteq \mathbb{K}^n$ such that

$$\Lambda = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{\zeta_i}.$$

- $\mathrm{rk} H_\Lambda = r$ and I_Λ is a radical ideal.

Theorem

Let $\Lambda \in R^*$ such that $\mathcal{A}_\Lambda$ is an r-dimensional $\mathbb{K}$-vector space. Then the following are equivalent:

- Up to $\mathbb{K}$-multiplication, there are r distinct common eigenvectors of $\{M_{x_i}^t\}_{i\in\{1,\dots,n\}}$.
- I_Λ is radical.

Let $f = -4y + 2z + 2yz + z^2$.
We know some entries of $\mathbb{H}_\Lambda$:

$$\mathbb{H}_\Lambda = \left(\begin{array}{c|cccccc} & 1 & y & z & y^2 & yz & z^2 \\ \hline 1 & f^*(1) & f^*(y) & f^*(z) & f^*(y^2) & f^*(yz) & f^*(z^2) \\ y & f^*(y) & f^*(y^2) & f^*(yz) & & & \\ z & f^*(z) & f^*(yz) & f^*(z^2) & & & \\ y^2 & f^*(y^2) & & & & & \\ yz & f^*(yz) & & & & ? & \\ z^2 & f^*(z^2) & & & & & \end{array}\right).$$

Let $f = -4y + 2z + 2yz + z^2$.

$$\mathbb{H}_\Lambda(\mathbf{h}) = \left(\begin{array}{c|cccccc} & 1 & y & z & y^2 & yz & z^2 \\ \hline 1 & 0 & -2 & 1 & 0 & 1 & 1 \\ y & -2 & 0 & 1 & h_{(3,0)} & h_{(2,1)} & h_{(1,2)} \\ z & 1 & 1 & 1 & h_{(2,1)} & h_{(1,2)} & h_{(0,3)} \\ y^2 & 0 & h_{(3,0)} & h_{(2,1)} & h_{(4,0)} & h_{(3,1)} & h_{(2,2)} \\ yz & 1 & h_{(2,1)} & h_{(1,2)} & h_{(3,1)} & h_{(2,2)} & h_{(1,3)} \\ z^2 & 1 & h_{(1,2)} & h_{(0,3)} & h_{(2,2)} & h_{(1,3)} & h_{(0,4)} \end{array}\right).$$

We want values for **h** such that $\mathrm{rk} H_\Lambda = r$ and I_Λ is radical.

Let $f = -4y + 2z + 2yz + z^2$.

$$\mathbb{H}_\Lambda(\mathbf{h}) = \left(\begin{array}{c|cccccc} & 1 & y & z & y^2 & yz & z^2 \\ \hline 1 & 0 & -2 & 1 & 0 & 1 & 1 \\ y & -2 & 0 & 1 & h_{(3,0)} & h_{(2,1)} & h_{(1,2)} \\ z & 1 & 1 & 1 & h_{(2,1)} & h_{(1,2)} & h_{(0,3)} \\ y^2 & 0 & h_{(3,0)} & h_{(2,1)} & h_{(4,0)} & h_{(3,1)} & h_{(2,2)} \\ yz & 1 & h_{(2,1)} & h_{(1,2)} & h_{(3,1)} & h_{(2,2)} & h_{(1,3)} \\ z^2 & 1 & h_{(1,2)} & h_{(0,3)} & h_{(2,2)} & h_{(1,3)} & h_{(0,4)} \end{array}\right).$$

We guess that $B = \{1, y, z\}$ is a basis for $\mathcal{A}_\Lambda$, so that $r = 3$. Define

$$\mathbb{H}_\Lambda^B = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let $f = -4y + 2z + 2yz + z^2$.

$$\mathbb{H}_\Lambda(\mathbf{h}) = \left(\begin{array}{c|cccccc} & 1 & y & z & y^2 & yz & z^2 \\ \hline 1 & 0 & -2 & 1 & 0 & 1 & 1 \\ y & -2 & 0 & 1 & h_{(3,0)} & h_{(2,1)} & h_{(1,2)} \\ z & 1 & 1 & 1 & h_{(2,1)} & h_{(1,2)} & h_{(0,3)} \\ y^2 & 0 & h_{(3,0)} & h_{(2,1)} & h_{(4,0)} & h_{(3,1)} & h_{(2,2)} \\ yz & 1 & h_{(2,1)} & h_{(1,2)} & h_{(3,1)} & h_{(2,2)} & h_{(1,3)} \\ z^2 & 1 & h_{(1,2)} & h_{(0,3)} & h_{(2,2)} & h_{(1,3)} & h_{(0,4)} \end{array}\right).$$

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$$\mathbb{H}^B_\Lambda = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbb{H}^B_{y\star\Lambda} = \begin{pmatrix} -2 & 0 & 1 \\ 0 & h_{(3,0)} & h_{(2,1)} \\ 1 & h_{(2,1)} & h_{(1,2)} \end{pmatrix}.$$

Let $f = -4y + 2z + 2yz + z^2$.
We guess that $B = \{1, y, z\}$ is a basis for $\mathcal{A}_\Lambda$, so that $r = 3$. Define

$$\mathbb{M}_y^B = \mathbb{H}_{y\star\Lambda}^B(\mathbb{H}_\Lambda^B)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{3}{8}h_{(3,0)} + \frac{1}{4}h_{(2,1)} & \frac{1}{8}h_{(3,0)} + \frac{1}{4}h_{(2,1)} & \frac{1}{4}h_{(3,0)} + \frac{1}{2}h_{(2,1)} \\ -\frac{3}{8}h_{(2,1)} + \frac{1}{4}h_{(1,2)} + \frac{1}{8} & \frac{1}{8}h_{(2,1)} + \frac{1}{4}h_{(1,2)} - \frac{3}{8} & \frac{1}{4}h_{(2,1)} + \frac{1}{2}h_{(1,2)} + \frac{1}{4} \end{pmatrix},$$

$$\mathbb{M}_z^B = \mathbb{H}_{z\star\Lambda}^B(\mathbb{H}_\Lambda^B)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{3}{8}h_{(2,1)} + \frac{1}{4}h_{(1,2)} + \frac{1}{8} & \frac{1}{8}h_{(2,1)} + \frac{1}{4}h_{(1,2)} - \frac{3}{8} & \frac{1}{4}h_{(2,1)} + \frac{1}{2}h_{(1,2)} + \frac{1}{4} \\ -\frac{3}{8}h_{(1,2)} + \frac{1}{4}h_{(0,3)} + \frac{1}{8} & \frac{1}{8}h_{(1,2)} + \frac{1}{4}h_{(0,3)} - \frac{3}{8} & \frac{1}{4}h_{(1,2)} + \frac{1}{2}h_{(0,3)} + \frac{1}{4} \end{pmatrix}.$$

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We want multiplication operators to commute!

$$\mathbb{M}_y^B\mathbb{M}_z^B - \mathbb{M}_z^B\mathbb{M}_y^B = 0.$$

$$\to h_{(3,0)} = -2,\ h_{(2,1)} = 1,\ h_{(2,1)} = 1,\ h_{(2,1)} = 4.$$

Let $f = -4y + 2z + 2yz + z^2$.

$$(\mathbb{M}_y^B)^t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \text{Eigenspaces: } \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

$$(\mathbb{M}_z^B)^t = \begin{pmatrix} 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{3}{4} \\ 1 & 1 & \frac{5}{2} \end{pmatrix} \longrightarrow \text{Eigenspaces: } \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\rangle.$$

Let $f = -4y + 2z + 2yz + z^2$.

Common eigenspaces: $\left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\rangle.$

Solve in λ_i : $f = \lambda_1 (1 - 1y + 0z)^2 + \lambda_2 (1 + 1y - \frac{1}{2}z)^2 + \lambda_3 (1 + 1y + 3z)^2$.

$$\lambda_1 = 1 \qquad \lambda_2 = -\frac{8}{7} \qquad \lambda_3 = \frac{1}{7}$$

Conclusion: $f = (1 - y)^2 - \frac{8}{7}(1 + y - \frac{1}{2}z)^2 + \frac{1}{7}(1 + y + 3z)^2$.

The algorithm
As proposed in [BCMT]

Algorithm: Symmetric tensor decomposition
Input: A homogeneous polynomial $f(x_0, x_1, \ldots, x_n)$ of degree d.
Output: A decomposition of f as $f = \sum_{i=1}^{r} \lambda_i \mathbf{k_i}(\mathbf{x})^d$ with r minimal.

- Compute the coefficients of f^*: $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$, for $|\alpha| \leq d$.
- $r := 1$.
- **repeat**
  1. Compute a set B of monomials of degree at most d connected to one with $|B| = r$.
  2. Find parameters $\mathbf{h}$ s.t. $\det(\mathbb{H}^B_\Lambda) \neq 0$ and the operators $\mathbb{M}_i = \mathbb{H}^B_{x_i \star \Lambda}(\mathbb{H}^B_\Lambda)^{-1}$ commute.
  3. If there is no solution, restart the loop with $r := r + 1$.
  4. Else compute the $n \times r$ eigenvalues $\zeta_{i,j}$ and the eigenvectors $\mathbf{v_j}$ s.t. $\mathbb{M}_i \mathbf{v_j} = \zeta_{i,j} \mathbf{v_j}$, $i = 1, \ldots, n, j = 1, \ldots, r$.

  until the eigenvalues are simple.
- Solve the linear system in $(l_j)_{j=1,\ldots,k}$: $\Lambda = \sum_{i=1}^{r} l_j \mathbb{1}_{\zeta_j}$ where $\zeta_i \in \mathbb{K}^n$ are the eigenvectors found in step 4.

Algorithm: Symmetric tensor decomposition
Input: A homogeneous polynomial $f(x_0, x_1, \ldots, x_n)$ of degree d
written by using a general set of essential variables.
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- ~~$r := 1$~~. r := #EssVar(f)?
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The refinements

1) The starting r

Algorithm: Symmetric tensor decomposition

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written by using a general set of essential variables.

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- Compute the coefficients of f^*: $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$, for $|\alpha| \leq d$.
- ~~$r := 1$~~. r := rk(Maximal numerical submatrix of H_Λ).
- **repeat**
  1. Compute a set B of monomials of degree at most d connected to one with $|B| = r$.
  2. Find parameters $\mathbf{h}$ s.t. $\det(\mathbb{H}_\Lambda^B) \neq 0$ and the operators $\mathbb{M}_i = \mathbb{H}_{x_i \star \Lambda}^B (\mathbb{H}_\Lambda^B)^{-1}$ commute.
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- Compute the coefficients of f^*: $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$, for $|\alpha| \leq d$.
- r := rk(largest numerical submatrix of H_Λ).
- **repeat**
  1. Compute a set B of monomials of degree at most d ~~connected to one~~ which is a complete staircase with $|B| = r$.
  2. Find parameters $\mathbf{h}$ s.t. $\det(\mathbb{H}_\Lambda^B) \neq 0$ and the operators $\mathbb{M}_i = \mathbb{H}_{x_i \star \Lambda}^B (\mathbb{H}_\Lambda^B)^{-1}$ commute.
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Connection to one: $B = \{1, y, y^2, y^2z, y^3\}$.
Complete staircase: $B = \{1, y, z, y^2, yz\}$.

Theorem

Let $F \in R$ be homogeneous written by using essential variables and let $\Lambda \in R^*$ be an extension of $f^* \in R^*_{\leq d}$. Then there is a monomial basis B of $\mathcal{A}_\Lambda$ such that B is a complete staircase.

Comparison with 3 variables

Size of B	# Complete staircases	# Connected to 1
3	1	5
4	3	13
5	5	35
6	9	96
7	13	267

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  until ~~the eigenvalues are simple.~~ there are r common eigenvectors.
- Solve the linear system in $(l_j)_{j=1,\ldots,k}$: $\Lambda = \sum_{i=1}^{r} l_j \mathbb{1}_{\zeta_j}$ where $\zeta_i \in \mathbb{K}^n$ are the eigenvectors found in step 4.

The refinements

3) Common eigenvectors

Example

$$F = (x+y)^3 + (x+z)^3 + (x+y+z)^3$$

$$\downarrow$$

$$\mathbb{M}_y^B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \quad \mathbb{M}_z^B = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\downarrow$$

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \quad \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle.$$

What we can learn more

Detecting the $I^{d-1}g$-case

Let $f := (x+y)^5 + (x+z)^5 + (x+2y)(x-y)^4$.
We check $r = 4$ and $B = \{1, y, z, y^2\}$.

$$\mathbb{M}_y^B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad \mathbb{M}_z^B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\downarrow$$

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Let $f := (x+y)^5 + (x+z)^5 + (x+2y)(x-y)^4$.
We check $r = 4$ and $B = \{1, y, z, y^2\}$.

$$\mathbb{M}_y^B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad \mathbb{M}_z^B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\downarrow$$

$$\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -5 \end{pmatrix} \right\rangle, \leftarrow \text{Generalized!}$$

$$\left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

Further work

- Get other (every?) configurations by studying Jordan blocks / type / ... ?
- What we can learn even more: the cactus rank.
- More selective choices on B? May these lead to bounds on r?
- How to deal with $\mathbf{h}$?
- Serious implementation? Complexity?

... and that is it.

Thanks for your attention!